Étude du modèle des variétés roulantes et de sa commandabilité

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THÈSE DE DOCTORAT

soutenue le 27/11/2012

par

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Study of the Rolling Manifolds Model and of its Controllability

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Résumé

Nous étudions la commandabilité du système de contrôle décrivant le procédé de roulement, sans glissement ni pivotement, de deux variétés riemanniennes n-dimensionnelles, l'une sur l'autre. Ce modèle est étroitement associé aux concepts de développement et d'holonomie des variétés, et il se généralise au cas de deux variétés affines. Les contributions principales sont celles données dans quatre articles, attachés à la fin de la thèse.

Le premier d'entre eux « Rolling manifolds and Controllability : the 3D case » traite le cas où les deux variétés sont 3-dimensionelles. Nous donnons alors, la liste des cas possibles pour lesquelles le système n'est pas commandable.

Dans le deuxième papier « Rolling manifolds on space forms », l'une des deux variétés est supposée être de courbure constante. On peut alors réduire l'étude de commandabilité à l'étude du groupe d'holonomie d'une certaine connexion vectorielle et on démontre, par exemple, que si la variété à courbure constante est une sphère n-dimensionelle et si ce groupe de l'holonomie n'agit pas transitivement, alors l'autre variété est en fait isométrique à la sphère.

Le troisième article « A Characterization of Isometries between Riemannian Manifolds by using Development along Geodesic Triangles » décrit, en utilisant le procédé de roulement (ou développement) le long des lacets, une version alternative du théorème de Cartan-Ambrose-Hicks, qui caractérise, entre autres, les isométries riemanniennes. Plus précisément, on prouve que si on part d'une certaine orientation initiale, et si on ne roule que le long des lacets basés au point initial (associé à cette orientation), alors les deux variétés sont isométriques si (et seulement si) les chemins tracés par le procédé de roulement sur l'autre variété, sont tous des lacets.

Finalement, le quatrième article « Rolling Manifolds without Spinning » étudie le procédé de roulement et sa commandabilité dans le cas où l'on ne peut pas pivoter. On caractérise alors les structures de toutes les orbites possibles en termes des groupes d'holonomie des variétés en question. On montre aussi qu'il n'existe aucune structure de fibré principal sur l'espace d'état tel que la distribution associée à ce modèle devienne une distribution principale, ce qui est à comparer notamment aux résultats du deuxième article.

Par ailleurs, dans la troisième partie de cette thèse, nous construisons soigneusement le modèle de roulement dans le cadre plus général des variétés affines, ainsi que dans celui des variétés riemanniennes de dimension différente.

Mots-clefs : Commandabilité, courbure, développement, géométrie (sous-)riemannienne, holonomie, orbite, variétés roulantes.

Study of the Rolling Manifolds Model and of its Controllability

Abstract

We study the controllability of the control system describing the rolling motion, without slipping nor spinning, of two n-dimensional Riemannian manifolds, one against the other. This model is closely related to the concepts of development and holonomy of the manifolds, and it generalizes to the case of affine manifolds. The main contributions are those given in four articles attached to the the thesis.

First of them "Rolling manifolds and Controllability: the 3D case" deal with the case where the two manifolds are 3-dimensional. We give the list of all the possible cases for which the system is not controllable.

In the second paper "Rolling manifolds on space forms" one of the manifolds is assumed to have constant curvature. We can then reduce the study of controllability to the study of the holonomy group of a certain vector bundle connection and we show, for example, that if the manifold with the constant curvature is an n-sphere and if this holonomy group does not act transitively, then the other manifold is in fact isometric to the sphere.

The third paper "A Characterization of Isometries between Riemannian Manifolds by using Development along Geodesic Triangles" describes, by using the rolling motion (or development) along the loops, an alternative version of the Cartan-Ambrose-Hicks Theorem, which characterizes, among others, the Riemannian isometries. More precisely, we prove that if one starts from a certain initial orientation, and if one only rolls along loops based at the initial point (associated to this orientation), then the two manifolds are isometric if (and only if) the paths traced by the rolling motion on the other manifolds, are all loops.

Finally, the fourth paper "Rolling Manifolds without Spinning" studies the rolling motion, and its controllability, when slipping is allowed. We characterize the structure of all the possible orbits in terms of the holonomy groups of the manifolds in question. It is also shown that there does not exist any principal bundle structure such that the related distribution becomes a principal distribution, a fact that is to be compared especially to the results of the second article.

Furthermore, in the third chapter of the thesis, we construct carefully the rolling model in the more general framework of affine manifolds, as well as that of Riemannian manifolds, of possibly different dimensions.

Keywords : Controllability, curvature, development, holonomy, orbit, rolling manifolds, (sub-)riemannian geometry.

Remerciements

Je tiens tout d'abord à exprimer ma reconnaissance la plus profonde à mon directeur de thèse, Yacine Chitour, pour ses conseils très profitables et indispensables durant ces trois années et demi de travail ensemble, sans lesquels cette thèse n'aurait jamais pu se réaliser. Sans aucun doute, j'ai appris énormément de son expérience et ses compétences mathématiques.

J'aimerais ensuite remercier très chaleureusement mes directeurs de thèse du côté finlandais, Markku Nihtilä et Jouko Tervo, pour ces nombreuses années de collaboration. C'est notamment grâce à eux que je me suis intéressé profondément aux mathématiques il y a environ dix ans, et ils m'ont toujours soutenu et encouragé.

Je voudrais dire un grand merci à Frédéric Jean pour son soutien et son aide au fil de ces dernières années. J'ai eu la chance de profiter de sa vaste expérience dans les domaines du contrôle géométrique et de la géométrie sous-riemannienne.

C'est notamment grâce aux personnes ci-dessus que j'ai eu le privilège de venir en France pour effectuer mes études doctorales à l'Université Paris-Sud sous la convention de cotutelle entre cette dernière et l'Université de l'est de la Finlande à Kuopio. Je vous en remercie tous.

Je suis très reconnaissant à Andrei Agrachev et Irina Markina qui ont accepté d'être les rapporteurs de la thèse ainsi qu'à Pierre Pansu de m'avoir fait l'honneur d'être le président du jury de ma soutenance. Je souhaite aussi exprimer toute ma gratitude aux autres membres du jury, Kirsi Peltonen et Knut Hüper.

D'autres personnes que j'aimerais remercier très chaleureusement pour ces dernières années passées ensemble sont, alphabétiquement, Davide Barilari, Ugo Boscain, Mauricio Godoy Molina, Paolo Mason, Gaspard Omnes, Marco Penna, Dario Prandi, Ludovic Rifford, Mario Sigalotti, Emmanuel Trélat, parmi d'autres.

Pour le support financier, mes remerciements les plus sincères vont à Finnish Academy of Science and Letters, KAUTE Foundation, Saastamoinen Foundation et l'Institut français de Finlande.

Je remercie l'Université Paris-Sud, le Laboratoire des Signaux et Systèmes et toute son équipe de m'avoir si bien accueilli.

Merci à tou(te)s mes ami(e)s, dont, bien entendu, les personnes mentionnées ci-dessus font partie, pour leur soutien et patience.

Finalement, je souhaiterais dédier cette thèse à ma famille et à Mari, dont le soutien a été immense pendant ces années. Je les remercie de tout mon cœur.

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Première partie

Introduction

Sommaire

1 Introduction

Dans cette thèse, nous étudions un modèle de roulement d'une variété différentielle sur une autre et quelques aspects de la commandabilité du système de contrôle associé. Le *procédé de roulement* (R) est sans *glissement* ni *pivotement*. D'ailleurs, si le glissement est permis, on appelle le procédé qui en résulte celui de *roulement sans pivotement* (NS) - (de l'anglais « No-Spinning »).

Quand les deux variétés sont isométriquement plongées dans un espace euclidien, le problème de roulement est classique en géométrie différentielle (voir [21]), à travers les notions de « développement d'une variété » et « d'application roulement »(« rolling map » en anglais). Pour se donner une idée intuitive du problème (une discussion plus sérieuse sur le sujet se trouve dans la section 8), considérons le problème de roulement d'une surface (bidimensionelle) convexe M sur une autre $\hat{M}$ surface dans l'espace euclidean $\mathbb{R}^3$, par exemple, le problème « plan-boule » (« plate-ball » en anglais), où une boule roule sur un plan dans $\mathbb{R}^3$ (voir [12, 14, 18]).

Les deux surfaces sont en contact, c.-à-d. elles ont un plan tangent commun au point de contact et, ce qui est (pratiquement) équivalent, leurs vecteurs normaux extérieurs sont opposés en ce point. Si $\gamma : [0, T] \to M$ est une courbe (disons lisse) sur M, la surface M est dite rouler sur $\hat{M}$ le long de γ sans glisser ni pivoter, si les conditions (SG) et (SP) présentées ci-dessous sont vérifiées.

Dans un premier temps, soient $\gamma : [0, T] \to M$, $\hat{\gamma} : [0, T] \to \hat{M}$ les chemins tracés sur M, $\hat{M}$, respectivement, par le point de contact. A l'instant $t \in [0, T]$, l'orientation relative (du plan tangent en $\hat{\gamma}(t)$) de $\hat{M}$ par rapport à (celui en $\gamma(t)$ de) M est mesurée par un angle $\theta(t)$ dans le plan tangent commun aux points de contact $\gamma(t)$, $\hat{\gamma}(t)$, respectivement. L'espace d'état Q du problème de roulement est alors de dimension cinq, puisque un point dans Q est défini en fixant un point sur M, un point sur $\hat{M}$ et un angle, c.-à-d. un point de S^1, le cercle unité (voir par exemple [2, 9]).

La condition de (SG) « roulement sans glissement » exige, pour tout $t \in [0, T]$, que la vitesse $\dot{\hat{\gamma}}(t)$ soit égale à la vitesse $\dot{\gamma}(t)$ tourné d'un angle $\theta(t)$. En revanche, la condition de (SP) « roulement sans pivotement » exige que les axes de rotation relatifs dans l'espace ambiant $\mathbb{R}^3$ des corps M et $\hat{M}$ restent dans le plan tangent commun, ce qui se traduit en une condition pour $\dot{\theta}(t)$.

Alors, une fois un point $\hat{x}_0$ sur $\hat{M}$ et un angle initial θ_0 sont choisis à l'instant $t = 0$, la courbe $\hat{\gamma}(t)$ et l'angle $\theta(t)$ sont uniquement déterminés, pour tout $t \in [0, T]$, par les conditions (SG)+(SP), et le procédé de roulement (R) en résulte. En ce qui concerne le roulement sans pivotement (NS) (où le glissement peut se produire), on choisit deux courbes (lisses) γ et $\hat{\gamma}$ sur M et $\hat{M}$, respectivement, ainsi qu'un angle initial θ_0 et on n'exige que seule la condition (SP) soit satisfaite. Il en résulte que l'orientation relative $\theta(t)$ est uniquement déterminée pour tout $t \in [0, T]$, et donc, on a bien une courbe dans Q décrivant le procédé de roulement sans pivotement.

Une question fondamentale associée au problème de roulement est celle de sa *commandabilité*, c'est-à-dire déterminer s'il existe, pour deux points donnés q_0, q_1 dans Q, une courbe γ dans M telle que la procédure de roulement de M sur $\hat{M}$ le long de γ amène le système de q_0 en q_1. Si c'est le cas, pour n'importe quels deux points q_0, q_1 dans Q, le modèle (ou le système de contrôle correspondant) est dit *complètement commandable*.

Si les variétés roulantes l'une sur l'autre sont de dimension deux, alors le problème de commandabilité est bien compris grâce aux travaux effectués dans [2], [6] et [16], parmi d'autres. Par exemple, dans le cas simplement connexe (et complet), le modèle de roulement est complètement commandable si et seulement si les variétés ne sont pas isométriques. En particulier, les ensembles atteignables sont des sous-variétés immergées de Q de dimension de soit 2, soit 5. Dans le cas où les surfaces sont convexes et isométriques, [16] donne une belle description de l'ensemble atteignable en dimension deux : considérons une configuration initiale pour les deux surfaces (convexes) en contact de sorte que l'une est une image miroir de l'autre par rapport au plan tangent (commun) situé au point de contact. Alors, cette propriété de position symétrique est conservée le long de roulement (R). Notons que pour deux surfaces isométriques, l'ensemble atteignable issu d'un point de contact où les courbures gaussiennes sont différentes, est en général ouvert (et donc, de dimension 5).

Du point de vue de la robotique, une fois la commandabilité est bien comprise, le problème suivant à traiter est celui de la *planification de mouvements* (« motion planning », en anglais), c.-à-d. déterminer une procédure efficace qui produit, pour toute paire de points (q_0, q_1) dans l'espace d'état Q, une courbe γ_{q_0,q_1} telle que le roulement de M sur $\hat{M}$ le long de cette courbe amène le système de q_0 en q_1. Dans [8], un algorithme basé sur une méthode de continuation a été proposé pour s'attaquer au problème de roulement d'une surface strictement convexe, compacte sur le plan euclidean de dimension 2. La convergence de cet algorithme a été démontrée dans [8] et c'était numériquement réalisé dans [1] (voir aussi [17] pour un autre algorithme).

Le modèle de roulement est traditionnellement présenté pour les variétés $M, \hat{M}$ isométriquement plongées dans un espace euclidien, généralement de dimension plus grande que celle des $M, \hat{M}$ (voir [10, 11, 21]), parce que c'est un cadre plus intuitif dans lequel parler des notions de pivotement et de glissement relatifs. Il s'avère toutefois que le modèle de roulement ne dépend que de la géométrie *intrinsèque* des variétés M et $\hat{M}$ (munies des métriques riemanniennes induites de l'espace euclidien ambiant). Par conséquent, le modèle de roulement peut être construit *intrinsèquement* et une fois que c'est fait, il est simple de généraliser le modèle pour toutes variétés riemanniennes (M, g), $(\hat{M}, \hat{g})$ de même dimension supérieure ou égale à deux.

La première étape vers une formulation intrinsèque du roulement, quand M et $\hat{M}$ sont orientées et leurs dimensions sont égales, disons $n \geq 2$, commence par une définition intrinsèque de l'espace d'état Q. Une orientation relative entre les deux variétés est représentée (par rapport aux repères orthonormaux donnés) par un élément de $\mathrm{SO}(n)$. Il en découle que la dimension de Q est $2n+n(n-1)/2$, car il est localement difféomorphe à $M \times \hat{M} \times \mathrm{SO}(n)$. Pour ce faire, il existe deux approches principales considérées pour la première fois dans [2] et [6]. Notons que ces deux références n'étudient que le cas bidimensionel (surfaces), mais les définitions d'espace d'état sont faciles à généraliser aux dimensions supérieures. Dans [2], l'espace d'état Q se compose de toutes les isométries infinitésimales entre tous les possibles plans tangents de M et $\hat{M}$ qui, de plus, respectent les orientations. Ceci est aussi la définition que nous allons adopter dans cette thèse.

La deuxième étape vers la formulation intrinsèque du roulement consiste à utiliser les transports parallèles, par rapport aux connexions de Levi-Civita, sur M et $\hat{M}$ (comme dans [2]) pour interpréter les contraintes de « sans pivotement » et « sans glissement » et à définir les trajectoires admissibles, c.-à-d. les courbes dans Q représentant le procédé de roulement (R). Cela nous mène à la construction d'une distribution n-dimensionelle $\mathcal{D}_{\mathrm{R}}$ sur Q telle que les courbes, disons absolument continues, tangentes à $\mathcal{D}_{\mathrm{R}}$ sont exactement les trajectoires admissibles pour le problème de roulement (voir [10]). Se posent alors les questions sur la commandabilité ainsi que sur la structure des ensembles atteignables (des orbites), du système commandé $(Q, \mathcal{D}_{\mathrm{R}})$, lesquelles constituent le thème principal de cette thèse.

Nous ne pouvons que trop insister sur le fait que le point de départ original de cette thèse est la construction d'un modèle de roulement intrinsèque général, pour des variétés riemanniennes quelconques. Le rôle-clé dans ce modèle est joué par la distribution de roulement $\mathcal{D}_{\mathrm{R}}$ qui capture les dynamiques de contrôle, ainsi que par le relèvement de roulement $\mathscr{L}_{\mathrm{R}}$ qui nous permet de relever les champs de vecteurs et les courbes de la variété de base M à l'espace d'état Q. Les définitions rigoureuses se trouvent dans la section 3, alors que leurs justifications (et généralisations) sont reportées à la partie III de la thèse.

2 Un aperçu de la thèse

Cette thèse se décline en quatre articles qui traitent de nombreuses questions de commandabilité liées au modèle de roulement (les articles A,B,C) et au modèle de roulement sans pivotement (l'article D). Par ailleurs, nous avons inclus la construction complète, et assez générale, de ces modèles de roulement, ainsi que la description de leurs propriétés de base.

Dans la section 3 nous commençons par introduire quelques notations et conventions générales ainsi que par rappeler le théorème de l'orbite de Sussmann. Nous définirons ensuite et brièvement le modèle de roulement et les concepts appropriés, notamment l'espace d'état $Q = Q(M, \hat{M})$ (Definition 3.9), le relèvement de roulement $\mathscr{L}_{\mathrm{R}}$ (Definition 3.5) et la distribution de roulement $\mathcal{D}_{\mathrm{R}}$ (Definition 3.5), et de même nous ferons quelques remarques sur certaines propriétés élémentaires nécessaires pour la partie II de la thèse. Les justifications supplémentaires ainsi que les preuves ont été omises de cette section et sont repoussées à la partie III.

Dans la partie II de la thèse, on décrit les principaux résultats des quatre articles sans donner de preuve et proposons quelques problèmes ouverts liés au sujet et aux résultats de chaque article décrit.

Plus précisément, la section 4 (article A) on donne la structure des orbites lorsque les variétés riemanniennes (M, g), $(\hat{M}, \hat{g})$ sont toutes les deux 3-dimensionelles. En particulier, il est montré que les dimensions possibles des orbites sont $3, 6, 7, 8, 9$.

La section 5 (article B) étudie le modèle du roulement et sa commandabilité dans le cas où $(\hat{M}, \hat{g})$ est de courbure constante $c \in \mathbb{R}$. Dans ce cadre, l'étude des ensembles atteignables se réduit à l'étude du groupe d'holonomie d'une certaine connexion (de roulement) ∇^c définie sur le fibré vectoriel $TM \oplus \mathbb{R} \to M$. Nous étudierons principalement les cas où $c = 0$ et $c = +1$, laissant le cas $c = -1$ comme un sujet de recherche ultérieur.

La section suivante 6 (article C) s'occupe d'un résultat de non commandabilité. On se demande quelle est la relation entre les géométries de (M, g) et $(\hat{M}, \hat{g})$ de même dimension si la condition suivante est vérifiée, à savoir qu'il existe deux points x_0 dans M

et $\hat{x}_0$ dans $\hat{M}$ ainsi qu'une orientation initiale A_0 en ces points, tels que, en roulant le long de n'importe quel lacet de M basé en x_0, le procédé de roulement ainsi engendré trace toujours un lacet sur $\hat{M}$ basé en $\hat{x}_0$. On prouve alors que les deux variétés riemanniennes sont (localement) isométriques.

Finalement, dans la section 7 (article D) on s'intéresse à la commandabilité du modèle de roulement sans pivotement, c.-à-d. que seul le glissement est permis. Il s'avère, par exemple, que la structure des orbites, et donc la réponse à la question de commandabilité, dépend uniquement des groupes d'holonomie de (M, g) et $(\hat{M}, \hat{g})$.

Dans la partie III, laquelle est écrite en anglais, nous construisons soigneusement le modèle de roulement d'abord pour des variétés affines puis riemanniennes. La section 8 introduit plus rigoureusement (par rapport à l'introduction) le modèle du roulement classique des surfaces (plongées) dans $\mathbb{R}^3$. On montrera aussi comment capturer les propriétés intrinsèques du modèle, ce qui sert à motiver sa généralisation du modèle aux variétés affines ou riemanniennes de dimensions quelconques. La section 9 introduit plus de notations lesquelles seront utiles pour montrer d'autres propriétés du modèle de roulement. Les sections 10 et 11 traitent de la définition et des propriétés de base du modèle de roulement pour, en premier lieu, des variétés affines et ensuite pour des variétés riemanniennes. Dans l'avant-dernière section 12 de la partie III, on caractérise l'intégrabilité de la distribution de roulement $\mathcal{D}_{\mathrm{R}}$. Enfin, la dernière section 13 de cette partie est consacrée au modèle du roulement sans glissement.

Après la partie III, se trouve la section des références. Nous n'essayons pas de donner une bibliographie exhaustive de la littérature sur le sujet, mais nous nous contentons plutôt de rester minimalistes : seuls les éléments dont cette thèse a strictement besoin sont cités. Plus de références peuvent être trouvées à la fin de chaque article.

Enfin, nous avons réuni dans la partie IV les quatre articles représentant la partie principale de cette thèse.

3 Notations et préliminaires

Dans cette thèse, les variétés différentielles sont toujours supposées lisses, séparées et à base dénombrable. Sauf mention explicite du contraire, toutes les applications, champs de vecteurs etc. sont, eux aussi, lisses. L'espace tangent et co-tangent d'une variétè M sont des espaces fibrés, notés $\pi_{TM} : TM \to M$, $\pi_{T^*M} : T^*M \to M$. Les autres fibrés tensoriels définis à partir de ceux-ci sont écrits avec des notations standard. On note $\mathrm{VF}(M)$ l'ensemble de champs de vecteurs sur M.

Si $\pi : E \to M$ est un espace fibré, sa fibre $\pi^{-1}(x)$ sur x sera notée $E|_x$. Dans le cas particulier de l'espace tangent et co-tangent, nous écrivons $T|_x M := (\pi_{TM})^{-1}(x)$, $T^*|_x M := (\pi_{T^*M})^{-1}(x)$. De plus, si $s : M \to E$ est une section d'un espace fibré $\pi : E \to M$, on écrit sa valeur $s|_x$ en $x \in M$ au lieu de $s(x)$. Par exemple, la valeur en $x \in M$ d'un $X \in \mathrm{VF}(M)$ est $X|_x$.

Pour une variété M, on note $\Omega_x(M)$ l'espace des lacets basés en $x \in M$, c.-à-d. les chemins $\gamma : [0, 1] \to M$ lisses par morceaux tels que $\gamma(0) = \gamma(1) = x$. La composition

$\gamma.\omega \in \Omega_x(M)$ de deux lacets $\gamma, \omega \in \Omega_x(M)$ est donnée par

$$\gamma.\omega(t) = \begin{cases} \omega(2t), & t \in [0, \frac{1}{2}] \\ \gamma(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Étant donné un fibré vectoriel $\pi : E \to M$ muni d'une connexion vectorielle ∇, nous écrivons R^∇ pour son tenseur de courbure, $(P^\nabla)^b_a(\gamma)$ pour le ∇-transport parallèle le long d'un chemin γ dans M de $\gamma(a)$ à $\gamma(b)$ et $H^\nabla|_x$ pour son groupe d'holonomie en $x \in M$.

Une variété affine (M, ∇) est composée d'une variété M est d'une connexion ∇ définie sur π_{TM}. Dans ce cadre, on note T^∇ le tenseur de torsion de ∇ et $\exp^\nabla_x$ l'application exponentielle de ∇ en $x \in M$. Dire que (M, ∇) est géodésiquement complet, signifie que l'application $\exp^\nabla_x$ est définie sur tout l'espace tangent $T|_x M$ en chaque point $x \in M$. Dans le cas où (M, g) est une variété riemannienne et ∇ est sa connexion de Levi-Civita, on dit que (M, g) est complet si (M, ∇) est géodésiquement complet.

Si $\mathcal{D} \subset TM$ est une distribution lisse et de rang constant sur M, on dira qu'un chemin $\gamma : I \to M$ absolument continu, où $I \subset \mathbb{R}$ est un intervalle non trivial, est *tangent à* $\mathcal{D}$, si $\dot{\gamma}(t) \in \mathcal{D}|_{\gamma(t)}$ pour presque tout $t \in I$. Pour $x \in M$, *l'orbite de* $\mathcal{D}$ *(où la* $\mathcal{D}$*-orbite) issue de* x est l'ensemble $\mathcal{O}_\mathcal{D}(x)$ défini par

$$\mathcal{O}_\mathcal{D}(x) = \{\gamma(1) \mid \gamma : [0,1] \to M \text{ absolument continu et tangent à } \mathcal{D} \text{ et } \gamma(0) = x\}.$$

On dit que $(M, \mathcal{D})$ est (ou $\mathcal{D}$ est) *(complètement) commandable* si $\mathcal{O}_\mathcal{D}(x) = M$ pour un (et donc pour tout) $x \in M$.

A cet effet, nous rappelons le *théorème de l'orbite.*

Théorème 3.1 (Sussmann, [3, 12, 22]) Soient $\mathcal{D}$ une distribution sur M et $x \in M$. Alors, $\mathcal{O}_\mathcal{D}(x)$ est une sous-variété immergée et connexe de M laquelle est, de plus, *faiblement plongée* dans le sens suivant : Si N est n'importe quelle variété et si $f : N \to M$ est lisse (resp. continue) telle que $f(N) \subset \mathcal{O}_\mathcal{D}(x)$, alors $f : N \to \mathcal{O}_\mathcal{D}(x)$ est lisse (resp. continue).

D'ailleurs, si $\mathcal{X} = \{X_i \mid X_i \in \mathrm{VF}(M), i \in I\}$ est une famille de champs de vecteurs (où I est un ensemble d'indices non vide) telle que $\{X|_x \mid X \in \mathcal{X}\}$ engendre $\mathcal{D}|_x$ pour tout $x \in M$, et si on note $(\Phi_X)_t(x)$ le flot de X à partir de x à l'instant t (s'il existe), alors

$$\mathcal{O}_\mathcal{D}(x) = \big\{ \big((\Phi_{X_{i_1}})_{t_1} \circ \cdots \circ (\Phi_{X_{i_k}})_{t_k}\big)(x) \bigm| k \in \mathbb{N},\ \{i_1, \dots, i_k\} \subset I,\ \{t_1, \dots, t_k\} \subset \mathbb{R}\big\}.$$

Remarque 3.2 Le théorème reste vrai pour les distributions lisses dont le rang n'est pas nécessairement constant.

Une variété riemannienne s'écrit (M, g), où g est la métrique riemannienne. La g-norme d'un vecteur $X \in TM$ est $\|X\|_g := \sqrt{g(X, X)}$.

Définition 3.3 Étant données deux variétés $M, \hat{M}$, on définit

$$T^*M \otimes T\hat{M} := \bigcup_{(x,\hat{x}) \in M \times \hat{M}} T^*|_x M \otimes T|_{\hat{x}} \hat{M}.$$

Les éléments de $T^*|_x M \otimes T|_{\hat{x}} \hat{M}$ s'identifient avec les applications $\mathbb{R}$-linéaires $T|_x M \to T|_{\hat{x}} \hat{M}$. Il est habituel dans cette thèse (ainsi que dans les articles A-D) d'écrire un point $A \in T^*M \otimes T\hat{M}$ comme $(x, \hat{x}; A)$ si $A \in T^*|_x M \otimes T|_{\hat{x}} \hat{M}$. Par ailleurs, la notation $q = (x, \hat{x}; A)$ pour un tel point insiste sur le fait que q est vu comme un point d'ensemble $T^*M \otimes T\hat{M}$ alors que A est considéré comme une application linéaire $T|_x M \to T|_{\hat{x}} \hat{M}$.

Nous définissons ensuite une distribution décrivant la condition de *roulement sans pivotement*, abrégée NS (de l'anglais « No-Spinning ») comme mentionnée dans l'introduction. Pour plus de justifications, nous faisons référence à la partie III (voir Définition 13.2 et Proposition 13.5).

Définition 3.4 Pour tous $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, et $X \in T|_x M$, $\hat{X} \in T|_{\hat{x}} \hat{M}$, on définit un vecteur tangent $\mathscr{L}_{\mathrm{NS}}(X, \hat{X})|_q$ de $T^*M \otimes T\hat{M}$ en q par

$$\mathscr{L}_{\mathrm{NS}}(X, \hat{X})|_q := \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \big((P^{\hat{\nabla}})_0^t(\hat{\gamma}) \circ A \circ (P^{\nabla})_t^0(\gamma)\big),$$

où γ et $\hat{\gamma}$ sont deux courbes lisses quelconques sur M et $\hat{M}$, respectivement, telles que $\dot{\gamma}(0) = X$, $\dot{\hat{\gamma}}(0) = \hat{X}$.

De plus, en chaque point $q \in T^*M \otimes T\hat{M}$, on considère un sous-espace $\mathcal{D}_{\mathrm{NS}}|_q$ de $T|_q(T^*M \otimes T\hat{M})$ tel que

$$\mathcal{D}_{\mathrm{NS}}|_q := \mathscr{L}_{\mathrm{NS}}(T|_x M \times T|_{\hat{x}} \hat{M})|_q,$$

et la distribution $\mathcal{D}_{\mathrm{NS}}$ sur $T^*M \otimes T\hat{M}$ définie par $q \mapsto \mathcal{D}_{\mathrm{NS}}|_q$, s'appelle *distribution de roulement sans pivotement* (généralisée) pour le procédé de roulement de (M, ∇) sur $(\hat{M}, \hat{\nabla})$.

La définition suivante, laquelle est un cas spécial de la précédente, est cruciale pour cette thèse, car on y introduit une distribution, $\mathcal{D}_{\mathrm{R}}$, qui décrit, à la fois, la condition de roulement sans glissement ainsi que la condition de roulement sans pivotement, un mouvement appelé *roulement* pour simplicité, comme discuté dans l'introduction.

Définition 3.5 Pour tous $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ et $X \in T|_x M$, on définit un vecteur tangent $\mathscr{L}_{\mathrm{R}}(X)|_q$ de $T^*M \otimes T\hat{M}$ en q par

$$\mathscr{L}_{\mathrm{R}}(X)|_q := \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \big((P^{\hat{\nabla}})_0^t(\hat{\gamma}) \circ A \circ (P^{\nabla})_t^0(\gamma)\big),$$

où γ et $\hat{\gamma}$ sont deux courbes lisses quelconques sur M et $\hat{M}$, respectivement, telles que $\dot{\gamma}(0) = X$, $\dot{\hat{\gamma}}(0) = AX$. L'application $X \mapsto \mathscr{L}_{\mathrm{R}}(X)|_q$ est appelée *relèvement du roulement* (en q).

De plus, nous définissons un sous-espace $\mathcal{D}_{\mathrm{R}}|_q$ de $T|_q(T^*M \otimes T\hat{M})$ par

$$\mathcal{D}_{\mathrm{R}}|_q := \mathscr{L}_{\mathrm{R}}(T|_x M)|_q.$$

La distribution $\mathcal{D}_{\mathrm{R}}$ sur $T^*M \otimes T\hat{M}$ donnée par $q \mapsto \mathcal{D}_{\mathrm{R}}|_q$ est appelée *distribution de roulement* (généralisée) pour le roulement de (M, ∇) sur $(\hat{M}, \hat{\nabla})$.

Remarque 3.6 Remarquons que pour tout point $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ et tout vecteur $X \in T|_x M$, on a

$$\mathscr{L}_{\mathrm{R}}(X)|_q = \mathscr{L}_{\mathrm{NS}}(X, AX)|_q,$$

et donc $\mathcal{D}_{\mathrm{R}}$ est une sous-distribution de $\mathcal{D}_{\mathrm{NS}}$.

Définition 3.7 Une courbe absolument continue $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [a, b]$, sur $T^*M \otimes T\hat{M}$, tangente à $\mathcal{D}_{\mathrm{R}}$, est appelée $\mathcal{D}_{\mathrm{R}}$*-relèvement de* γ *passant par* $q(a) = q_0$, et est notée

$$q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = \big(\gamma(t), \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t); A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)\big).$$

Remarque 3.8 Il sera montré dans la partie III (Proposition 10.7) que pour chaque $q_0 = (x_0, \hat{x}_0; A_0) \in T^*M \otimes T\hat{M}$ et pour chaque courbe absolument continue $\gamma : [0, 1] \to M$ telle que $\gamma(0) = x_0$, il existe un $\mathcal{D}_{\mathrm{R}}$-relèvement unique de γ passant par q_0, défini sur un intervalle $[0, T]$, $0 \leq T \leq 1$.

Pour les variétés riemanniennes de dimension égale, le concept suivant d'un espace d'état sera employé dans la partie II. Il sera motivé (et généralisé) dans la section 8 (et dans la définition 11.1) de la partie III.

Définition 3.9 Soient $(M, g), (\hat{M}, \hat{g})$ des variétés riemanniennes connexes, orientées et de même dimension $\dim M = \dim \hat{M}$. *L'espace d'état* pour le roulement de (M, g) sur $(\hat{M}, \hat{g})$ est l'ensemble

$$\begin{aligned} Q(M, \hat{M}) := \{A \in T^*|_x M \otimes T|_{\hat{x}} \hat{M} \mid (x, \hat{x}) \in M \times \hat{M},\ \det A > 0 \\ \|AX\|_{\hat{g}} = \|X\|_g\,,\ \forall X \in T|_x M\}. \end{aligned}$$

Remarque 3.10 On va justifier dans la partie III que $\mathcal{D}_{\mathrm{NS}}$ et $\mathcal{D}_{\mathrm{R}}$ sont des distributions lisses et de rangs constants sur $T^*M \otimes T\hat{M}$, ainsi que le fait qu'elles se restreignent pour devenir des distributions lisses de mêmes rang sur $Q(M, \hat{M})$.

Par ailleurs, dans la section 11 nous allons introduire l'espace $Q(M, \hat{M})$ dans le cadre plus général où $M, \hat{M}$ sont de dimension différente et non orientées.

Définition 3.11 On définit

$$\begin{aligned} \pi_{T^*M \otimes T\hat{M}} &: T^*M \otimes T\hat{M} \to M \times \hat{M}; \quad (x, \hat{x}; A) \mapsto (x, \hat{x}) \\ \pi_{T^*M \otimes T\hat{M}, M} &: T^*M \otimes T\hat{M} \to M; \quad (x, \hat{x}; A) \mapsto x \\ \pi_{T^*M \otimes T\hat{M}, \hat{M}} &: T^*M \otimes T\hat{M} \to \hat{M}; \quad (x, \hat{x}; A) \mapsto \hat{x}. \end{aligned}$$

Si, de plus, (M, g), $(\hat{M}, \hat{g})$ sont des variétés riemanniennes connexes, orientées et de même dimension, nous écrivons

$$\begin{aligned} \pi_{Q(M,\hat{M})} &:= \pi_{T^*M \otimes T\hat{M}}|_{Q(M,\hat{M})} : Q(M, \hat{M}) \to M \times \hat{M} \\ \pi_{Q(M,\hat{M}),M} &:= \pi_{T^*M \otimes T\hat{M},M}|_{Q(M,\hat{M})} : Q(M, \hat{M}) \to M \\ \pi_{Q(M,\hat{M}),\hat{M}} &:= \pi_{T^*M \otimes T\hat{M},\hat{M}}|_{Q(M,\hat{M})} : Q(M, \hat{M}) \to \hat{M}. \end{aligned}$$

Nous allons conclure cette section avec les concepts suivants, qui jouent un rôle très important dans les articles A and C (à cet effet, voir aussi le remarque qui suit la définition).

Définition 3.12 Pour tout $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, on définit la *courbure de roulement* $R^{\mathsf{Rol}}|_q$ *en* q comme

$$R^{\mathsf{Rol}}|_q : T|_x M \wedge T|_x M \to T^*|_x M \otimes T|_{\hat{x}}\hat{M};$$
$$R^{\mathsf{Rol}}|_q(X,Y)Z := A(R^{\nabla}(X,Y)Z) - R^{\hat{\nabla}}(AX,AY)AZ$$

et la *torsion* $T^{\mathsf{Rol}}|_q$ *de roulement en* q comme

$$T^{\mathsf{Rol}}|_q : T|_x M \wedge T|_x M \to T|_{\hat{x}}\hat{M};$$
$$T^{\mathsf{Rol}}|_q(X,Y) := AT^{\nabla}(X,Y) - T^{\hat{\nabla}}(AX,AY).$$

Remarque 3.13 Les notations ci-dessus proviennent de l'article D. En ce qui concerne les articles A et C, nous y avons employé, respectivement, les notations Rol_q et $\mathcal{R}_A^{(\nabla,\hat{\nabla})}$ au lieu de $R^{\mathsf{Rol}}|_q$ en $q = (x, \hat{x}; A)$.

Deuxième partie

Principaux résultats

Sommaire

4 Roulement des variétés riemanniennes de dimension 3

Le but de cette section est de décrire les principaux résultats de l'article A, qui traite de la commandabilité du modèle de roulement dans le cas où (M, g), $(\hat{M}, \hat{g})$ sont connexes, orientées et de dimension 3. Nous notons simplement Q l'espace d'état $Q(M, \hat{M})$.

Dans la suite, on aura besoin des notations suivantes.

Définition 4.1 (i) Soit (N, h) une variété riemannienne, $I \subset \mathbb{R}$ un intervalle ouvert et $f \in C^\infty(I)$ qui ne s'annule en aucun point de I. Alors, le *produit tordu* de I et N par rapport à la *fonction de distorsion* f est $(I \times N, \mathrm{d}r^2 \oplus_f h)$, où r est la coordonnée naturelle sur I (induite de $\mathbb{R}$), et la métrique $\mathrm{d}r^2 \oplus_f h$ sur $I \times N$ est telle que sa valeur est $ab + f(r)^2 h(X|_x, Y|_x)$ sur les vecteurs $a\partial_r + X, b\partial_r + Y \in T|_{(r,x)}(I \times M)$. Ici et dans la suite, ∂_r est le champs de coordonnées naturel (positivement dirigé) sur I.

(ii) Étant donné un $\beta > 0$, nous disons qu'une variété riemmannienne (N, h) de dimension 3 *appartient à une classe* $\mathcal{M}_\beta$, s'il existe un repère orthonormé (E_1, E_2, E_3) et des fonctions $c, \gamma_1, \gamma_3 \in C^\infty(N)$ tels que les relations de commutation suivantes soient satisfaites :

$$\begin{aligned} [E_1, E_2] &= cE_3 \\ [E_2, E_3] &= cE_1 \\ [E_3, E_1] &= -\gamma_1 E_1 + 2\beta E_2 - \gamma_3 E_3. \end{aligned}$$

Un tel repère est dit *adapté*.

La non-commandabilité de $(Q, \mathcal{D}_{\mathrm{R}})$ dans ce cadre tridimensionel, ainsi que les dimensions possibles des orbites, est décrit *essentiellement complètement* par les théorèmes suivants.

Théorème 4.2 Soient (M, g), $(\hat{M}, \hat{g})$ des variétés riemanniennes tridimensionelles, connexes et orientées. Si pour un point $\tilde{q} \in Q$ l'orbite $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\tilde{q})$ n'est pas ouverte dans Q, alors il existe un sous-ensemble ouvert et dense O de $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\tilde{q})$ tel qu'en chaque point $q_0 = (x_0, \hat{x}_0; A_0) \in O$ correspondent des voisinages $U \subset M$, $\hat{U} \subset \hat{M}$ de x_0 et $\hat{x}_0$, respectivement, pour lesquelles l'un des cas suivants est vrai :

(i) Il y a une isométrie $\phi : (U, g) \to (\hat{U}, \hat{g})$ telle que $\phi_*|_{x_0} = A_0$;

(ii) (U, g) et $(\hat{U}, \hat{g})$ sont des variétés appartenantes à la classe $\mathcal{M}_\beta$, pour un $\beta > 0$.

(iii) (U, g) et $(\hat{U}, \hat{g})$ sont des produits tordus de la forme $(U, g) = (I \times N, \mathrm{d}r^2 \oplus_f h)$, $(\hat{U}, \hat{g}) = (I \times \hat{N}, \mathrm{d}\hat{r}^2 \oplus_{\hat{f}} \hat{h})$ où $I \subset \mathbb{R}$ est un intervalle ouvert , (N, h), $(\hat{N}, \hat{h})$ sont des variétés riemanniennes quelconques et $f, \hat{f} \in C^\infty(I)$ satisfont l'une des conditions suivantes :

(a) Soit $A_0\partial_r|_{x_0} = \partial_{\hat{r}}|_{\hat{x}_0}$ et $\dfrac{f'(t)}{f(t)} = \dfrac{\hat{f}'(t)}{\hat{f}(t)}$, pour $t \in I$,

(b) soit il existe une constante $K \in \mathbb{R}$ telle que $\dfrac{f''(t)}{f(t)} = -K = \dfrac{\hat{f}''(t)}{\hat{f}(t)}$ pour tout $t \in I$.

Remarque 4.3 La condition $A_0\partial_r|_{x_0} = \partial_{\hat{r}}|_{\hat{x}_0}$ dans (iii)-(a), qui n'est pas parue dans la formulation du théorème 5.1 cas (c)-(A) dans l'article A, a été ajoutée afin de mieux correspondre au cas (ii)-(b1) du théorème suivant. Cela suit de la proposition 5.28 de l'article A.

Un théorème inverse au précédent est formulé ensuite. Il inclut aussi de l'information sur les dimensions possibles des orbites. Rappelons que $\dim Q = 9$.

Théorème 4.4 Soient (M, g), $(\hat{M}, \hat{g})$ des variétés riemanniennes de dimension 3, connexes et orientées, $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ et on suppose que $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ n'est pas une variété intégrale de $\mathcal{D}_{\mathrm{R}}$. Définissons des ouverts $M(q_0) := \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $\hat{M}(q_0) := \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, de M, $\hat{M}$, respectivement. Alors, nous avons les résultats suivants :

(i) Soient $\beta > 0$ et (M, g), $(\hat{M}, \hat{g})$ appartenant à la classe $\mathcal{M}_\beta$ avec les repères adaptés (E_1, E_2, E_3) et $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$, respectivement.
- (a) Si $A_0E_2|_{x_0} = \pm\hat{E}_2|_{\hat{x}_0}$, alors $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 7$.
- (b) Supposons que $A_0E_2|_{x_0} \neq \pm\hat{E}_2|_{\hat{x}_0}$. Alors, $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$, sauf si l'une seulement des deux variétés $(M(q_0), g)$ ou $(\hat{M}(q_0), \hat{g})$ est de courbure constante, et dans ce cas $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 7$.

(ii) Supposons que $(M, g) = (I \times N, \mathrm{d}r^2 \oplus_f h)$, $(\hat{M}, \hat{g}) = (\hat{I} \times \hat{N}, \mathrm{d}\hat{r}^2 \oplus_{\hat{f}} \hat{h})$ sont des produits tordus, où $I, \hat{I} \subset \mathbb{R}$ est un intervalle ouvert, et écrivons $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$.

(a) Si $A_0\partial_r|_{x_0} = \partial_{\hat{r}}|_{\hat{x}_0}$ et si

$$\frac{f'(t + r_0)}{f(t + r_0)} = \frac{\hat{f}'(t + \hat{r}_0)}{\hat{f}(t + \hat{r}_0)}$$

est vrai pour chaque t tel que $(t + r_0, t + \hat{r}_0) \in I \times \hat{I}$, alors $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$.

(b) Soit $K \in \mathbb{R}$ et soient $f, \hat{f}$ telles que

$$\frac{f''(r)}{f(r)} = -K = \frac{\hat{f}''(\hat{r})}{\hat{f}(\hat{r})}, \quad \forall (r, \hat{r}) \in I \times \hat{I}.$$

- (b1) Si $A_0\partial_r|_{x_0} = \pm\partial_{\hat{r}}|_{\hat{x}_0}$ et $\frac{f'(r_0)}{f(r_0)} = \pm\frac{\hat{f}'(\hat{r}_0)}{\hat{f}(\hat{r}_0)}$ où les possibilités $\pm$ correspondent dans les deux cas, alors $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$.
- (b2) Si (seulement) une des $(M(q_0), g)$ ou $(\hat{M}(q_0), \hat{g})$ est de courbure constante, alors $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$.
- (b3) Dans tous les autres cas, $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

Comme un corollaire aux deux théorèmes précédents, nous avons une liste de toutes les dimensions possibles des orbites non ouvertes (c.-à-d. celles dont les dimensions sont $< 9 = \dim Q$).

Corollaire 4.5 Si (M, g), $(\hat{M}, \hat{g})$ sont tridimensionelles et si une orbite $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ est non ouverte dans Q, alors

$$\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \in \{3, 6, 7, 8\}.$$

Par ailleurs, toutes les quatre dimensions possibles dans le membre de droite sont réalisées.

Remarque 4.6 (1) Une variété riemannienne (N, h) est dite *sasakienne* s'il existe un champ de Killing ξ de longueur unité sur N tel que la courbure riemannienne R de (N, h) satisfasse

$$R(X, \xi)Y = h(\xi, Y)X - h(X, Y)\xi, \quad \forall X, Y \in \mathrm{VF}(N).$$

Le champ de Killing ξ est appelé *champs caractéristique* de la variété sasakienne (M, g). Voir [7] (Proposition 1.1.2 cas (ii)).
Il est facile de démontrer que si (M, g) appartient à la classe $\mathcal{M}_\beta$, $\beta > 0$, ayant le repère adapté (E_1, E_2, E_3), alors $(M, \beta^2 g)$ est sasakienne et βE_2 est son champ caractéristique. En revanche, si (M, g) est sasakienne, alors pour tout $\beta > 0$, l'espace $(M, \beta^{-2} g)$ appartient *localement* à la classe $\mathcal{M}_\beta$ (c.-à-d. chaque $x \in M$ a un voisinage ouvert U tel que $(U, \beta^2 g)$ appartient à la classe $\mathcal{M}_\beta$).
Il n'est alors pas difficile d'étendre le cas (i) du théorème ci-dessus de sorte qu'on puisse y remplacer les variétés appartenant à la classe $\mathcal{M}_\beta$ avec les variétés sasakiennes. L'avantage dans cette extension est l'élégance : le concept d'une variété sasakienne est défini d'une façon plus invariante que celui d'une variété appartenant à la classe $\mathcal{M}_\beta$.

(2) Les variétés de contact de dimension 3 sont caractérisées par deux invariants κ, χ définis sur ces variétés, voir [4]. Là encore, il n'est pas difficile de démontrer que les variétés appartenant à la classe $\mathcal{M}_\beta$, $\beta > 0$, sont des variétés de contact telles que $\chi = 0$ et, inversement, une variété de contact avec $\chi = 0$ peut être munie d'une métrique riemanninenne qui la rende une variété appartenant localement à la classe $\mathcal{M}_\beta$, $\beta > 0$ (voir l'article A).

Problèmes ouverts

1. Que peut-on dire de la structure *globale* de (M, g) et de $(\hat{M}, \hat{g})$ dans le théorème 4.2 ?

   Par exemple, on sait (les remarques similaires s'appliquant à $(\hat{M}, \hat{g})$), par le théorème, que l'ouvert $M(q_0)$ (comme défini dans le théorème 4.4) de M au-dessous de l'orbite $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ contient un ensemble ouvert et dense V qui, localement, soit appartient à la classe $\mathcal{M}_\beta$ pour un $\beta > 0$, soit est un produit tordu ayant une fonction de distorsion de type bien défini. Or, on peut démontrer que si V' est un sous-ensemble ouvert de V qui en même temps appartient à la classe $\mathcal{M}_\beta$ et est un produit tordu du type décrit ci-dessus, alors (V', g) est de courbure constante β^2.

   Est-il possible de construire une variété riemannienne (M, g) contenant deux ouverts M_1, M_2 (resp. trois ouverts M_1, M_2, M_3) tels que $M_1 \cup M_2$ (resp. $M_1 \cup M_2 \cup M_3$) est dense dans M et (M_1, g) appartient à la classe $\mathcal{M}_\beta$ alors que (M_2, g) est un produit tordu du type comme ci-dessus (resp. et, de plus, (M_3, g) est de courbure constante β^2) ?

2. Il est probable qu'on puisse prouver des analogues des théorèmes 4.2 et 4.4 dans les cas où $\dim M = 2$, $\dim \hat{M} = 3$ ou $\dim M = 3$, $\dim \hat{M} = 2$. La dimension de l'espace d'état $Q(M, \hat{M})$ (comme introduit dans la section 11) est 8 dans les deux cas, et l'on s'attend à ce que le dernier cas soit plus facile à traiter puisque on y dispose de trois contrôles alors que dans le premier cas, on n'en a que deux.

3. Prouver un résultat de classification similaire lorsque $\dim M = \dim \hat{M} = 4$. Pour cette dimension, les variétés riemanniennes (M, g), $(\hat{M}, \hat{g})$ ont des structures supplémentaires (par exemple, les opérateurs de Hodge dans $\bigwedge^2 TM$ et $\bigwedge^2 T\hat{M}$) qui pourraient être utiles (cf. [20], Chapter 24).

5 Roulement sur un espace de courbure constante

Dans cette section, nous supposons que $(\hat{M}, \hat{g})$ est une variété riemannienne simplement connexe, complète et de courbure constante $c \in \mathbb{R}$. Un tel espace est uniquement déterminé, à une isométrie près, par le nombre réel c. De plus, les deux variétés connexes et orientées (M, g), $(\hat{M}, \hat{g})$ sont de même dimension n et on écrit simplement Q pour $Q(M, \hat{M})$. Ce qui suit est l'objet principal de l'article B.

Définissons tout d'abord un groupe de Lie connexe qui joue un rôle fondamental dans ce cadre.

Définition 5.1 Soit $n \in \mathbb{N}$. On pose $G_0(n) := \mathrm{SE}(n)$ et pour $c \in \mathbb{R}$, $c \neq 0$, soit $G_c(n)$ la composante de l'identité du groupe des automorphismes linéaires de $\mathbb{R}^{n+1}$ qui laissent invariante la forme bi-linéaire $\langle \cdot, \cdot \rangle_{n;c}$ donnée par

$$\langle x, y \rangle_{n;c} := \sum_{k=1}^{n} x_k y_k + c^{-1} x_{n+1} y_{n+1},$$

où $x = (x_1, \ldots, x_{n+1}), y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$.

Remarque 5.2 Remarquons que $G_1(n) = \mathrm{SO}(n+1)$, tandis que $G_{-1}(n) = \mathrm{SO}_0(n, 1)$, la composante de l'identité de $\mathrm{SO}(n, 1)$. Par ailleurs, pour $c > 0$ (resp. $c < 0$) $G_c(n)$ est isomorphe à $G_1(n)$ (resp. $G_{-1}(n)$). Donc, dans la famille des groupes $G_c(n)$, $c \in \mathbb{R}$, il n'y a que trois éléments non isomorphe : $G_1(n) = \mathrm{SO}(n+1), G_0(n) = \mathrm{SE}(n)$ et $G_{-1}(n) = \mathrm{SO}_0(n, 1)$.

Il est bien connu que l'ensemble des isométries préservant l'orientation d'un espace $(\hat{M}, \hat{g})$ simplement connexe, complète et de courbure constante c, est isomorphe à $G_c(n)$. On déduit de cette observation un résultat (voir Proposition 4.1 dans l'article B), qui est important dans l'analyse du système de contrôle $(Q, \mathcal{D}_{\mathrm{R}})$ dans la situation présente.

Proposition 5.3 Soit $(\hat{M}, \hat{g})$ un espace simplement connexe, complet et de courbure constante c. Alors, les propositions suivantes sont vraies :

(i) Il existe une action de groupe $\mu : G_c(n) \times Q \to Q$ sur Q qui rend $\pi_{Q,M} : Q \to M$ un $G_c(n)$-fibré principal à gauche et $\mathcal{D}_{\mathrm{R}}$ une connexion principale de ce fibré, c.-à-d. $(\mu_B)_* \mathcal{D}_{\mathrm{R}}|_q = \mathcal{D}_{\mathrm{R}}|_{\mu(B,q)}$ pour tout $(B, q) \in G_c(n) \times Q$. On a écrit ici $\mu_B : Q \to Q$; $\mu_B(q) = \mu(B, q)$.

(ii) En chaque point $q = (x, \hat{x}; A)$, il existe un sous-groupe unique $\mathcal{H}_q^c$ de $G_c(n)$, appelé *groupe d'holonomie de* $\mathcal{D}_{\mathrm{R}}$ *en* q, tel que

$$\mu(\mathcal{H}_q^c \times \{q\}) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \cap \pi_{Q,M}^{-1}(x).$$

De plus, tous les groupes d'holonomie $\mathcal{H}_q^c$, $q \in Q$, sont conjugués.

Ceci veut dire que l'étude de la commandabilité (ou bien de la non-commandabilité) de $(Q, \mathcal{D}_{\mathrm{R}})$ se réduit à l'étude des groupes d'holonomie $\mathcal{H}_q^c$, $q \in Q$, dans le sens où $(Q, \mathcal{D}_{\mathrm{R}})$ est commandable si et seulement si $\mathcal{H}_q^c = G_c(n)$ en un (et donc en chaque) point $q \in Q$. Par ailleurs, toutes les orbites de $\mathcal{D}_{\mathrm{R}}$ sont difféomorphes l'une à l'autre par l'action μ.

On se débarrasse du paramètre c en multipliant les métriques des espaces (M, g) et $(\hat{M}, \hat{g})$ par la même constante (qui est $|c|$ si $c \neq 0$ et 1 si $c = 0$) et il suffit alors de considérer les cas où $c \in \{-1, 0, +1\}$.

En utilisant la théorie standard des espace fibrés associés et des connexions linéaires dans le cadre des fibrés principaux munis de connections principales (cf. [13]), on peut prouver le théorème suivant, qui est le théorème 4.5 dans l'article B (quand $c \neq 0$).

Théorème 5.4 Soit $\pi_{TM \oplus \mathbb{R}} : TM \oplus \mathbb{R} \to M$ un fibré vectoriel tel que $(X, r) \mapsto x$ si $X \in T|_x M$. Pour tout $c \in \mathbb{R}$, on définit une connexion linéaire ∇^c sur $\pi_{TM \oplus \mathbb{R}}$ en posant

$$\nabla^c_Y(X, r) := (\nabla_Y X + rY, Y(r) - cg(X, Y)), \quad \forall X, Y \in \mathrm{VF}(M),\ r \in C^\infty(M).$$

Alors, pour chaque $q = (x, \hat{x}; A) \in Q$, le groupe d'holonomie $H^{\nabla^c}|_x$ de ∇^c en x est isomorphe à $\mathcal{H}_q^c$. De plus, si pour un $c \neq 0$ on définit un produit scalaire h_c sur $TM \oplus \mathbb{R}$ par

$$h_c((X, r), (Y, s)) := g(X, Y) + c^{-1} rs,$$

alors ∇^c est métrique par rapport à h_c, c.-à-d. pour tous $X, Y, Z \in \mathrm{VF}(M), r, s \in C^\infty(M)$, on a

$$Z\big(h_c((X, r), (Y, s))\big) = h_c(\nabla^c_Z(X, r), (Y, s)) + h_c((X, r), \nabla^c_Z(Y, s)).$$

Remarque 5.5 La connexion ∇^c a été définie dans l'article B seulement dans le cas où $c \neq 0$ et on l'appelle *connexion de roulement* ∇^{Rol}.

Il n'est pas difficile de voir que la preuve du théorème dans l'article B, après quelques petites modifications, marche aussi quand $c = 0$ (même si la définition de h_c pour $c = 0$ n'a pas de sens).

Nous pouvons formuler maintenant les théorèmes principaux de l'article B, qui s'occupent de la question de commandabilité du système de roulement quand $(\hat{M}, \hat{g})$ est complète, simplement connexe et de courbure constante, soit $c = 0$ (c.-à-d. le plan n-dimensionnel), soit $c = +1$ (c.-à-d. la sphère unité n-dimensionnelle). Commençons par une formulation du théorème 4.3 de l'article B.

Théorème 5.6 Soit (M, g) une variété riemannienne complète, connexe, orientée, de dimension $n \geq 2$ et soit $(M, g) = \mathbb{R}^n$, le plan euclidean de dimension n. Alors, le problème de roulement est complètement commandable si et seulement si le groupe d'holonomie de (M, g) est égal à $\mathrm{SO}(n)$ (à un isomorphisme près).

Le résultat ci-dessus répond complètement à la question de commandabilité dans le cas où $(M, g) = \mathbb{R}^n$. Quant à la commandabilité du problème de roulement quand $(\hat{M}, \hat{g})$ est la sphère unité de dimension n, on a le résultat suivant partiel, qui est le théorème 4.6 dans l'article B.

Théorème 5.7 Soit (M, g) une variété riemannienne complète, simplement connexe, orientée et de dimension $n \geq 2$, et soit $(\hat{M}, \hat{g}) = S^n$, la sphère unité standard de dimension n. Si en un (et donc en chaque) point $x \in M$, le groupe d'holonomie $H^{\nabla^1}|_x$ de ∇^1 n'agit pas transitivement sur la sphère unité de $(T|_x M \oplus \mathbb{R}, h_1|_{T|_x M \oplus \mathbb{R}})$, alors (M, g) admet S^n comme revêtement universel riemannien. En particulier, le problème de roulement n'est pas complètement commandable dans ce cas.

En regardant la liste de sous-groupes connexes et fermés de $\mathrm{SO}(n)$ (établie par Berger, voir [13], Section 3.4.3), qui agissent transitivement sur la sphère unité de $\mathbb{R}^n$, on déduit un corollaire du théorème précédent (Corollaire 4.7 dans l'article B).

Corollaire 5.8 Soit (M, g) comme dans le théorème précédent et $(\hat{M}, \hat{g}) = S^n$. Si n est paire et $n \geq 16$, alors le problème de roulement est complètement commandable si et seulement si (M, g) n'est pas de courbure constante égale à 1.

Problèmes ouverts

1. Classifier toutes les variétés riemanniennes (M, g) pour lesquelles $(Q, \mathcal{D}_{\mathrm{R}})$ n'est pas complètement commandable, quand $(\hat{M}, \hat{g})$ est simplement connexe, complète et de courbure constante $c \neq 0$. Le théorème 5.6 est un résultat partiel dans cette direction, quand $c = +1$ (et donc quand $c > 0$).
2. Classifier tous les groupes d'holonomie de ∇^c, pour $c \neq 0$.
3. Sans supposer que (M, g) soit simplement connexe et complète, prouver pour $c \neq 0$ (ou $c \in \{-1, +1\}$) un résultat local analogue au théorème de de Rham : si $H^{\nabla^c}|_x$, en un point $x \in M$, agit d'une façon *réductible* sur $T|_x M \oplus \mathbb{R}$ (c.-à-d. il laisse invariant un sous-espace linéaire non trivial V de $T|_x M \oplus \mathbb{R}$), alors que peut-on dire de la structure locale de (M, g) dans un voisinage de ce point x ?
4. Est-ce que le résultat inverse de la proposition 5.3 est vrai ? Plus précisément, si $\pi_{Q,M}$ est muni d'une structure de G-fibré principal, pour un groupe de Lie G, est-il vrai que $(\hat{M}, \hat{g})$ doit être de courbure constante ?
5. Il serait aussi intéressant d'étudier la commandabilité dans le cas où $(\hat{M}, \hat{g})$ est un espace symétrique. Par exemple, dans ce cadre, la structure de crochets de Lie des champs de vecteurs tangents à $\mathcal{D}_{\mathrm{R}}$ est considérablement simplifiée.

6 Caractérisation des isométries riemanniennes en roulant le long des lacets

Cette section fait l'objet de l'article C dans lequel on s'intéresse au problème de roulement d'une variété riemannienne (M, g) le long des lacets, basés en un point x_0 fixé, sur une autre variété riemannienne $(\hat{M}, \hat{g})$ sous l'hypothèse que la courbe sur $\hat{M}$ engendrée par le procédé de roulement à partir d'un point initial $q_0 = (x_0, \hat{x}_0; A_0)$ dans Q, est un lacet dans $\hat{M}$ basé en $\hat{x}_0$. Il s'avère que, si l'on suppose que les deux variétés $(M, g), (\hat{M}, \hat{g})$ sont de même dimension n, complètes, simplement connexes et orientées, alors il existe une isométrie $\phi : (M, g) \to (\hat{M}, \hat{g})$ satisfaisant $\phi_*|_{x_0} = A_0$. Le résultat principal pourrait être vu, dans un certain sens, comme une version du théorème

Cartan-Ambrose-Hicks (voir Théorème 12.1). Nous allons noter $Q(M, \hat{M})$ simplement par Q.

En fait, on a un peu plus, car on n'a pas besoin de faire le roulement le long de tous les lacets dans M basés en un point donné x_0, mais il suffit de le faire le long des lacets définis par deux triangles géodésiques attachés en x_0.

Définition 6.1 Un lacet $\gamma \in \Omega_x(M)$ dans une variété affine (M, ∇) est appelé *triangle géodésique* basé en x s'il existe $0 = t_0 < t_1 < t_2 < t_3 = 1$ tels que $\gamma|_{[t_i, t_{i+1}]}$ soit une géodésique pour tout $i = 0, 1, 2$. Notons $\triangle_x(M, \nabla)$ l'ensemble des triangles géodésiques basés en $x \in M$ et

$$\triangle_x^2(M, \nabla) := \{\omega.\gamma \mid \omega, \gamma \in \triangle_x(M, \nabla)\},$$

l'ensemble des compositions de deux triangles géodésiques basés en x.

Le théorème principal 3.1 de l'article C est le suivant.

Théorème 6.2 Soient (M, g), $(\hat{M}, \hat{g})$ des variétés riemanniennes complètes, connexes, orientées et de même dimension n et supposons aussi que M est simplement connexe. Alors, étant donné un point $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, il existe un revêtement riemannien $\phi : (M, g) \to (\hat{M}, \hat{g})$ tel que $\phi_*|_{x_0} = A_0$ si et seulement si

$$\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0) \in \Omega_{\hat{x}_0}(\hat{M}), \quad \forall \gamma \in \triangle_{x_0}^2(M, \nabla). \tag{1}$$

Ici, ∇ est la connection de Levi-Civita de (M, g).

Remarque 6.3 La preuve du théorème est basée sur un résultat technique que nous allons nous rappeler dans un moment et qui est formulé pour des variétés affines (M, ∇), $(\hat{M}, \hat{\nabla})$ de dimension éventuellement différente. Cela fait aussi usage du théorème de Cartan-Ambrose-Hicks.

Nous n'allons pas répéter les détails ici mais il faut faire une remarque sur les notations. D'après la démonstration de la proposition 10.7 ci-dessous, pour tout $q = (x, \hat{x}; A) \in Q$ et pour toute courbe absolument continue $\gamma : [0, 1] \to M$ telle que $\gamma(0) = x$, on a

$$\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q)(t) = (\Lambda_{\hat{x}}^{\hat{\nabla}})^{-1}\big(A \circ \Lambda_x^{\nabla}(\gamma)\big)(t), \quad \forall t \in [0, 1],$$

ce qui était écrit comme $\Lambda_A^{(\nabla, \hat{\nabla})}(\gamma)(t)$ dans l'article C.

Remarque 6.4 On peut remplacer la condition (1) dans le théorème précédent par une condition plus forte,

$$\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0) \in \Omega_{\hat{x}_0}(\hat{M}), \quad \forall \gamma \in \Omega_{x_0}(M).$$

Pour les détails, et pour une liste plus étendue des versions alternatives du théorème de Cartan-Ambrose-Hicks en géométrie riemannienne, voir Théorème 5.2 dans l'article C.

Pour conclure cette section, on formule le résultat technique (Proposition 4.1 dans l'article C) qu'on utilise, avec le théorème de Cartan-Ambrose-Hicks, dans la preuve du théorème ci-dessus. La notation a été adaptée pour qu'elle corresponde à celle utilisée partout dans cette thèse.

Proposition 6.5 Soient $(M, \nabla), (\hat{M}, \hat{\nabla})$ des variétés affines (éventuellement de dimension différente) et soit $q_0 = (x_0, \hat{x}_0; A_0) \in T^*M \otimes T\hat{M}$. Soient $U \subset T|_{x_0}M$, $\hat{U} \subset T|_{\hat{x}_0}\hat{M}$ des domaines de définition des applications exponentielles $\exp^{\nabla}_{x_0}$, $\exp^{\hat{\nabla}}_{\hat{x}_0}$, respectivement, et notons $\gamma_u(t) := \exp^{\nabla}_{x_0}(tu)$, $\hat{\gamma}_{\hat{u}}(t) := \exp^{\hat{\nabla}}_{\hat{x}_0}(t\hat{u})$ pour $u \in U$, $\hat{u} \in \hat{U}$, $t \in [0, 1]$. Alors, si

$$\begin{aligned}
&\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0) \in \triangle_{\hat{x}_0}(\hat{M}, \hat{\nabla}), \quad \forall \gamma \in \triangle_{x_0}(M, \nabla) \text{ t.q. } \exists q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1), \\
&T^{\mathsf{Rol}}|_{q_{\mathcal{D}_{\mathrm{R}}}(\gamma_u, q_0)(1)}(\dot{\gamma}_u(1), \cdot) = 0, \quad \forall u \in U \cap A_0^{-1}(\hat{U}),
\end{aligned}$$

on a pour tout $u \in U \cap A_0^{-1}(\hat{U})$ que

$$\begin{aligned}
&(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0 u} \circ A_0 = A_{\mathcal{D}_{\mathrm{R}}}(\gamma_u, q_0)(1) \circ (\exp^{\nabla}_{x_0})_*|_u \\
&R^{\mathsf{Rol}}|_{q_{\mathcal{D}_{\mathrm{R}}}(\gamma_u, q_0)(1)}(\dot{\gamma}_u(1), \cdot)\dot{\gamma}_u(1) = 0.
\end{aligned}$$

Remarque 6.6 La condition $\exists q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1)$ veut juste dire qu'on suppose que $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)$ est définie sur $[0, 1]$, qui est l'intervalle de définition de γ. Nous avons, de plus, fait usage du fait que $\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma_u, q_0) = \hat{\gamma}_{A_0 u}$, quand $u \in U \cap A_0^{-1}(\hat{U})$.

Problèmes ouverts

1. Peut-on remplacer l'hypothèse " $\forall \gamma \in \triangle^2_{x_0}(M, \nabla)$ " dans l'équation (1) du théorème 6.2 par " $\forall \gamma \in \square_{x_0}(M, \nabla)$ ", où $\square_{x_0}(M, \nabla)$ est l'ensemble composé des *quadrilatères géodésiques* basés en x_0, voir Définition 2.1 (et Remarque 5.3, cas (c)) dans l'article C ?
2. Peut-on généraliser, peut-être après avoir remplacé $\triangle^2_{x_0}(M, \nabla)$ dans l'équation (1) par un ensemble de lacets plus grand, le théorème 3.1 de l'article C (c.-à-d. le théorème 6.2 ci-dessus) au cas où (M, g) et $(\hat{M}, \hat{g})$ sont de dimension différente ? Que pourrions-nous dire dans le cas de deux variétés affines (M, ∇), $(\hat{M}, \hat{\nabla})$ (outre la proposition 6.5) ?

7 Roulement sans pivotement

Dans la présente section, nous résumons les résultats principaux de l'article D, dans lequel on étudie le modèle de roulement où le glissement est permis mais non le pivotement. Nous commençons par le cas des variétés affines et ensuite nous nous restreignons au cas riemannien. Les espaces d'état appropriés restent les mêmes qu'auparavant, c'est-à-dire $T^*M \otimes T\hat{M}$ ou Q, mais maintenant, à la place de la distribution de roulement $\mathcal{D}_{\mathrm{R}}$ de rang $n = \dim M$, nous nous concentrons sur une distribution $\mathcal{D}_{\mathrm{NS}}$ (et le système de contrôle associé) de rang $n + \hat{n}$, où $\hat{n} = \dim \hat{M}$, contenant $\mathcal{D}_{\mathrm{R}}$.

Il faut observer que l'article D traite de la situation plus générale de fibrés vectoriels au lieu de variétés affines (ou riemanniennes), mais pour la raison d'unification de cet exposé on se contente, dans ce qui suit, du cas moins général. L'espace $Q(M, \hat{M})$ sera noté simplement Q.

Contrairement à la distribution de roulement $\mathcal{D}_{\mathrm{R}}$, les (fibres des) orbites de la distribution $\mathcal{D}_{\mathrm{NS}}$ de roulement sans pivotement sont faciles à décrire en termes des groupes d'holonomie de ∇ et $\hat{\nabla}$ (voir Proposition 3.13 dans l'article D).

Proposition 7.1 Supposons que (M,∇), $(\hat{M},\hat{\nabla})$ sont des variétés affines et soit $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$. Alors, la fibre au-dessus de $(x,\hat{x})$ d'une orbite $\mathcal{O}_{\mathcal{D}_{\mathrm{NS}}}(q)$ de $\mathcal{D}_{\mathrm{NS}}$ est donnée par

$$(\pi_{T^*M\otimes T\hat{M}})^{-1}(x,\hat{x})\cap\mathcal{O}_{\mathcal{D}_{\mathrm{NS}}}(q)=H^{\hat{\nabla}}|_{\hat{x}}\circ A\circ H^{\nabla}|_x,$$

avec $H^{\hat{\nabla}}|_{\hat{x}}\circ A\circ H^{\nabla}|_x:=\{\hat{B}\circ A\circ B\mid \hat{B}\in H^{\hat{\nabla}}|_{\hat{x}},\ B\in H^{\nabla}|_x\}$.

Avant de se restreindre au cas riemannien, nous donnons, comme corollaire au résultat précédent, une condition nécessaire et suffisante pour qu'une $\mathcal{D}_{\mathrm{NS}}$-orbite soit une variété intégrale de $\mathcal{D}_{\mathrm{NS}}$ (voir Corollaires 3.10 at 3.12 ainsi que Remarque 3.14 dans l'article D).

Corollaire 7.2 Notons pour $x\in M$ et $\hat{x}\in\hat{M}$, $\mathfrak{h}^{\nabla}|_x$ et $\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}}$, respectivement, les algèbres de Lie des $H^{\nabla}|_x$ et $H^{\hat{\nabla}}|_{\hat{x}}$. Étant donné $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$, on a que $\mathcal{O}_{\mathcal{D}_{\mathrm{NS}}}(q)$ est une variété intégrale de $\mathcal{D}_{\mathrm{NS}}$ si et seulement si

$$\mathrm{im}\mathfrak{h}^{\nabla}|_x\subset\ker A\quad\text{and}\quad\mathrm{im}A\subset\ker\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}}.$$

Remarque 7.3 La condition $\mathrm{im}\mathfrak{h}^{\nabla}|_x\subset\ker A$ (resp. $\mathrm{im}A\subset\ker\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}}$) signifie que $A\circ U=0$ pour tout $U\in\mathfrak{h}^{\nabla}|_x$ (resp. $\hat{U}\circ A=0$ pour tout $\hat{U}\in\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}}$).

Corollaire 7.4 Si $n=\hat{n}$ et si $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$ est tel que $A:T|_xM\to T|_{\hat{x}}\hat{M}$ est inversible, alors $\mathcal{O}_{\mathcal{D}_{\mathrm{NS}}}(q)$ est une variété intégrale de $\mathcal{D}_{\mathrm{NS}}$ si et seulement si (M,∇) et $(\hat{M},\hat{\nabla})$ sont plates.

Dans le reste de la section, nous nous restreignons au cas où (M,g), $(\hat{M},\hat{g})$ sont des variétés riemanniennes, connexes, orientées, de même dimension n et on note ∇, $\hat{\nabla}$, respectivement, leurs connexions de Levi-Civita.

Aussi tentant que puisse paraître le fait que la fibre typique de π_Q est difféomorphe à $\mathrm{SO}(n)$, afin d'espérer qu'il existe une structure de $\mathrm{SO}(n)$-fibré principal sur π_Q, le théorème suivant, avec lequel on devrait comparer Proposition 5.3, nous dit qu'en général, quand $n\geq 3$, il n'y a pas de telle structure qui rende $\mathcal{D}_{\mathrm{NS}}$ connexion principale (voir Théorème 4.12 dans l'article D).

Théorème 7.5 Supposons que $n\geq 3$. Soit $(x_0,\hat{x}_0)\in M\times\hat{M}$ donné et identifions $H^{\nabla}|_{x_0}$, $H^{\hat{\nabla}}|_{\hat{x}_0}$ aux sous-groupes $H,\hat{H}$, respectivement, de $\mathrm{SO}(n)$ par rapport aux repères g- et $\hat{g}$-orthonormaux quelconques en x_0 et $\hat{x}_0$. Alors, si $H\cap\hat{H}$ n'est pas un sous-groupe fini de $\mathrm{SO}(n)$, il en découle que π_Q n'a aucune structure de fibré principal dont l'action laisserait $\mathcal{D}_{\mathrm{NS}}$ invariante.

En particulier, ceci est vrai si l'un des H ou $\hat{H}$ est $\mathrm{SO}(n)$, alors que l'autre n'est pas fini.

Remarque 7.6 Si $n = \hat{n} = 2$, il est bien connu (voir par exemple [2, 3, 9]) que π_Q porte une structure naturelle d'un SO(2)-fibré principal. Dans ce cas, on peut montrer que la distribution $\mathcal{D}_{\mathrm{NS}}$ est en effet invariante par rapport à l'action donnant cette structure.

Remarque 7.7 Il est aussi bien connu que le groupe d'holonomie d'une variété riemannienne « générique » de dimension n est (isomorphe à) SO(n). Si c'était le cas, il découlerait du théorème précédent que H est un groupe fini, et donc $(\hat{M}, \hat{g})$ est plate.

Ceci suggère qu'une structure de SO(n)-fibré principal qui, de plus, laisse $\mathcal{D}_{\mathrm{NS}}$ invariante, puisse exister si $(\hat{M}, \hat{g})$ est plate. On va montrer plus tard que cela est vrai, du moins si $(\hat{M}, \hat{g})$ est le plan euclidien de dimension n. Cependant, parce que ce résultat n'est pas donné dans l'article D, nous en fournissons une preuve à la section 13 (Proposition 13.11).

On conclut cette section avec deux résultats sur la commandabilité de $(Q, \mathcal{D}_{\mathrm{NS}})$ (voir Proposition 4.11 et Théorème 4.12 dans l'article D).

Proposition 7.8 Supposons que $M, \hat{M}$ sont simplement connexes. Étant donné $(x_0, \hat{x}_0) \in M \times \hat{M}$ et n'importe quels repères g- et $\hat{g}$-orthonormaux F en x_0 et $\hat{F}$ en $\hat{x}_0$, identifions, par rapport aux F et $\hat{F}$, les algèbres de Lie des $H^{\nabla}|_{x_0}$, $H^{\hat{\nabla}}|_{\hat{x}_0}$ aux sous-algèbgres de Lie $\mathfrak{h}$, $\hat{\mathfrak{h}}$ de $\mathfrak{so}(n)$. Alors, $(Q, \mathcal{D}_{\mathrm{NS}})$ est complètement commandable si et seulement si

$$\mathfrak{h} + \hat{\mathfrak{h}} = \mathfrak{so}(n).$$

Théorème 7.9 Supposons que (M, g), $(\hat{M}, \hat{g})$ sont des variétés riemanniennes, complètes, simplement connexes, non symétriques, irréductibles, orientées et de dimension $n \geq 2$ où $n \neq 8$. Soient $x_0 \in M$, $\hat{x}_0 \in \hat{M}$ arbitraires. Alors, $\mathcal{D}_{\mathrm{NS}}$ est complètement commandable dans Q si et seulement si $H^{\nabla}|_{x_0}$ ou $H^{\hat{\nabla}}|_{\hat{x}_0}$ est égal à SO(n) (par rapport aux repères orthonormaux quelconques en x_0 et $\hat{x}_0$).

Part III

Models of Rolling Manifolds

Summary

8 Classical Rolling Model in $\mathbb{R}^3$

In this section we recall the rolling model of two oriented smooth, connected, embedded surfaces $M, \hat{M} \subset \mathbb{R}^3$ and use it to justify the more general model of rolling that will be the subject of the next section. We restrict here in such low dimensions for simplicity; For the rolling model of k-dimensional sub-manifold of $\mathbb{R}^n$, where $n \geq k \geq 2$, and for similar considerations, we refer to [10, 21]. What follows is a partial repetition of section 4.3 in article D.

Let us write for $N, \hat{N}$ for some choice of unit normal vector fields of $M, \hat{M}$, respectively. We are considering the model where the surface M rolls on $\hat{M}$ without *slipping* nor *spinning*. The ingredients of such a model are the *configuration space* (i.e. the state space) and *dynamics* (here a control system) of such a motion. Here the configuration space $Q_{\mathbb{R}^3}(M, \hat{M})$ should consist of all the possible ways to make the surface M to touch tangentially $\hat{M}$ in a uniquely determined manner. In order to do that, the minimal amount of information needed are the desired, respective *points of contact* $x \in M$, $\hat{x} \in \hat{M}$ and some rigid motion $(U, a) \in \mathrm{SE}(3)$ which moves M in such a way that x moves to $\hat{x}$ and the (affine) tangent plane of M at x moves to coincide with the (affine) tangent plane of $\hat{M}$ at $\hat{x}$. It is clear that this amounts to requiring that

$$UN|_x = \pm \hat{N}|_{\hat{x}}, \quad a = \hat{x} - x,$$

where '$\pm$' depends on from "which side" M touches $\hat{M}$ after the rigid motion. Up to changing N to $-N$, we may assume that the '$+$'-case takes place here.

Notice that the above conditions don't uniquely determine (U, a) since if $\hat{S}$ is any rotation about the axis $\hat{N}|_{\hat{x}}$, then $(\hat{S}U, a)$ satisfies the above condition as well. However, the translational part a of the rigid motion (U, a) is uniquely determined by the above relation. This justifies the following definition (see Definition 4.16 in D).

Definition 8.1 The space of admissible *contact configurations*, or *state space*, for rolling of M against $\hat{M}$ is

$$Q_{\mathbb{R}^3}(M, \hat{M}) := \{(x, \hat{x}, U) \in M \times \hat{M} \times \mathrm{SO}(3) \mid UN|_x = \hat{N}|_{\hat{x}}\}.$$

Remark 8.2 The condition $UN|_x = \hat{N}|_{\hat{x}}$ prevents M intersection $\hat{M}$ at contact points at *infinitesimal level only* and we allow in this model possible intersections of $M, \hat{M}$ outside of these contact points. If the reader considers this to be implausible, (s)he can restrict here to think only about the case where M, $\hat{M}$ are convex (closed) surfaces with N an outward unit vector field and $\hat{N}$ an inward unit vector field (or vice-versa).

The above definition is *a priori* extrinsic since it uses the normal vector fields $N, \hat{N}$ which don't emerge from the intrinsic (Riemannian) geometries of $M, \hat{M}$. We let, from now on, $g, \hat{g}$ to be the Riemannian metrics of $M, \hat{M}$, respectively, induced by the embeddings of them in $\mathbb{R}^3$ from the usual inner product of $\mathbb{R}^3$. These metrics in hand, the space $Q_{\mathbb{R}^3}(M, \hat{M})$ can be intrinsically characterized as shown in the next lemma (see D, Lemma 4.17).

Lemma 8.3 Defining

$$Q(M,\hat{M}) := \{A \in T^*|_x M \otimes T|_{\hat{x}}\hat{M} \mid (x,\hat{x}) \in M \times \hat{M},$$
$$\det(A) > 0,\ \|AX\|_{\hat{g}} = \|X\|_g,\ \forall X \in T|_x M\},$$

then the map

$$Q_{\mathbb{R}^3}(M,\hat{M}) \to Q(M,\hat{M}); \quad (x,\hat{x},U) \mapsto U|_{T|_x M}$$

is a bijection.

Notice that $\det(A) > 0$ or $\det(A) < 0$ makes sense since M and $\hat{M}$ are assumed to be oriented. Moreover, the linear map $U|_{T|_x M} : T|_x M \to \mathbb{R}^3$ corresponding to $(x,\hat{x},U) \in Q_{\mathbb{R}^3}(M,\hat{M})$ can be seen as a map $U|_{T|_x M} : T|_x M \to T|_{\hat{x}}\hat{M}$ since if $X \in T|_x M$ i.e. $\langle X, N|_x\rangle = 0$, then $\left\langle UX, \hat{N}|_{\hat{x}}\right\rangle = \langle UX, UN|_x\rangle = \langle X, N|_x\rangle = 0$ i.e. $UX \in T|_{\hat{x}}\hat{M}$.

Remark 8.4 The definition of the space $Q(M,\hat{M})$ in the above lemma makes sense for any (abstract) Riemannian manifolds (M,g), $(\hat{M},\hat{g})$ of equal dimension $n \geq 2$ and hence readily generalizes the notion of state space for rolling in higher dimensions.

Our next task is to describe reasonable dynamics in the context of the rolling model. The first approach will be again *a priori* extrinsic but, as we will see, it can be characterized purely intrinsically.

To begin with, we introduce for every $z \in \mathbb{R}^3$ the linear map J_z by

$$J_z : \mathbb{R}^3 \to \mathbb{R}^3; \quad J_z(y) := z \times y,$$

where '$\times$' is the cross product operation. Observe that $J_z \in \mathfrak{so}(3)$. Moreover, in the Lie algebra $\mathfrak{so}(3)$ of SO(3) we use the inner product

$$\langle A, B\rangle_{\mathfrak{so}} := -\mathrm{tr}(AB), \quad A, B \in \mathfrak{so}(3).$$

Suppose now that one is given an initial contact configuration $q_0 = (x_0,\hat{x}_0,U_0) \in Q_{\mathbb{R}^3}(M,\hat{M})$ and a curve $\hat{\gamma} : [0,\hat{T}] \to \hat{M}$ in $\hat{M}$, $\hat{T} > 0$, such that $\hat{\gamma}(0) = \hat{x}_0$. One wishes to roll M against $\hat{M}$ along $\hat{\gamma}$, starting from the configuration q_0, in such a way that:

(1) The respective contact points $\gamma(t)$ on M, with $\gamma(0) = x_0$, generated by the rolling motion move with the same relative velocity as those, i.e. $\hat{\gamma}(t)$, on $\hat{M}$;

(2) The relative axis of rotation of M at the moment t is parallel to the tangent space of $\hat{M}$ at $\hat{\gamma}(t)$.

Let $q(t) = (\gamma(t),\hat{\gamma}(t),U(t))$, $t \in [0,T]$, be a curve in the contact configuration space $Q_{\mathbb{R}^3}(M,\hat{M})$. Then the *relative velocity* of γ at time t is nothing more that $U(t)\dot{\gamma}(t)$ and condition (1) dictates that this should equal $\dot{\hat{\gamma}}(t)$. On the other hand, the relative axis of rotation of M at the moment t is the unique quantity $z(t) \in \mathbb{R}^3$ such that $J_{z(t)} = \dot{U}(t)U(t)^{-1}$ (where $\dot{U}(t)U(t)^{-1}$ is the relative speed of rotation of M) and condition (2) means that $\left\langle z(t), \hat{N}|_{\hat{\gamma}(t)}\right\rangle = 0$ for all t. These observations along with the fact that for all $y, z \in \mathbb{R}^3$, $\langle y, z\rangle = \langle J_y, J_z\rangle_{\mathfrak{so}}$ permit us to give a precise mathematical formulation of the rolling dynamics, which was formally described in (1)-(2) above (see D, Definition 4.18). By convention, we reverse, with respect to what was done above, the roles of γ and $\hat{\gamma}$ so as to actually generate the curve $\hat{\gamma}$ on $\hat{M}$ starting from a curve γ in M.

Definition 8.5 Given $q_0 = (x_0, \hat{x}_0, U_0) \in Q_{\mathbb{R}^3}$ and a curve $\gamma : [0,T] \to M$ in M, $T > 0$, such that $\gamma(0) = x_0$, we say that M rolls against $\hat{M}$ along γ starting from (initial contact configuration) q_0 without *slipping* nor *spinning* if and only if there exists a curve $(\hat{\gamma}(t), U(t))$, $t \in [0,T]$, in $\hat{M} \times \mathrm{SO}(3)$ so that the following conditions are satisfied for all t:

(i) Contact: $U(t)N|_{\gamma(t)} = \hat{N}|_{\hat{\gamma}(t)}$;
(ii) No-slipping: $U(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$;
(iii) No-spinning: $\left\langle \dot{U}(t)U(t)^{-1}, J_{\hat{N}|_{\hat{\gamma}(t)}} \right\rangle_{\mathfrak{so}} = 0$.

The condition (ii) is called the *no-slipping* condition, while the condition (iii) is called the *no-spinning condition*.

Finally, the intrinsic characterization of conditions (ii)-(iii) is given by the next proposition (see Proposition 4.22 in paper D). We use ∇ and $\hat{\nabla}$ to denote the Levi-Civita connections of (M, g), $(\hat{M}, \hat{g})$, respectively.

Proposition 8.6 Let $q(t) = (\gamma(t), \hat{\gamma}(t), U(t))$ be a curve in $Q_{\mathbb{R}^3}(M, \hat{M})$ and let $A(t) := U(t)|_{T|_{\gamma(t)}}$ be the corresponding curve in $Q(M, \hat{M})$ provided by Lemma 8.3 above. Then $q(t)$ satisfies conditions (ii)-(iii) in Definition 8.5 (condition (i) being superfluous) for all t if and only if the following hold for all t

(I) No-slipping: $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$,
(II) No-spinning: $A(t)\nabla_{\dot{\gamma}(t)}X(t) = \hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(t)X(t))$ for all vector fields X along γ.

Remark 8.7 The above equations (I) and (II) make sense in the case of any (abstract) affine manifolds (M, ∇), $(\hat{M}, \hat{\nabla})$ of any dimensions, provided that the condition on $A(t)$ to be in $Q(M, \hat{M})$ is relaxed to that of $A(t)$ being only in, say, $T^*M \otimes T\hat{M}$.

In view of Lemma 8.3 and Proposition 8.6, which provide, respectively, intrinsic (w.r.t Riemannian geometry) characterizations of the state space and the dynamics of the rolling problem, there is no difficulty to extend the rolling model in the case of two Riemannian manifolds (M, g), $(\hat{M}, \hat{g})$ of the same dimension. In the next section we will do so eventually, but we will, in view of the last Remark, start with even more general case of affine manifolds (M, ∇), $(\hat{M}, \hat{\nabla})$ of possibly different dimensions.

9 Some Notations and Preliminary Results

We will collect to this section some additional notations and results that will be used in the sections that follow in this part of the thesis.

Write $\|\cdot\|_{\mathbb{R}^n}$, for $n \in \mathbb{N}$, for the the standard Euclidean norm in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ for the standard Euclidean inner product in $\mathbb{R}^n$. We call the vector $e_i \in \mathbb{R}^n$ the i-th standard basis vector if its i-th entry is 1 and the rest are zero. Also, $e_1, \dots, e_n$ is called the standard basis of $\mathbb{R}^n$.

Definition 9.1 For $k, m \in \mathbb{N}$, define $\mathcal{L}_k(\mathbb{R}^m)$ to be the space of all linear maps $\mathbb{R}^k \to \mathbb{R}^m$ and $O_k(\mathbb{R}^m)$ to be the set of all $B \in \mathcal{L}_k(\mathbb{R}^m)$ such that

(i) if $k \leq m$, then $\|Bu\|_{\mathbb{R}^m} = \|u\|_{\mathbb{R}^k}$ for all $u \in \mathbb{R}^k$;
(ii) if $k \geq m$, then B is surjective and $\|Bu\|_{\mathbb{R}^m} = \|u\|_{\mathbb{R}^k}$ for all $u \in (\ker B)^\perp$.
Here $S^\perp$ denotes the orthogonal complement of $S \subset \mathbb{R}^k$ with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}^k}$.

The next proposition is standard. We omit the proof.

Proposition 9.2 Let $k, m \in \mathbb{N}$. Then
(i) $O_k(\mathbb{R}^m)$ is a closed smooth submanifold of $\mathcal{L}_k(\mathbb{R}^m)$;
(ii) the map $\mathcal{L}_k(\mathbb{R}^m) \to \mathcal{L}_m(\mathbb{R}^k)$; $A \mapsto A^T$, where A^T is the usual transpose of A, restricts to a diffeomorphism $O_k(\mathbb{R}^m) \to O_m(\mathbb{R}^k)$;
(iii) if $k \neq m$, then $O_k(\mathbb{R}^m)$ is connected. On the other hand, $O_k(\mathbb{R}^k)$ is diffeomorphic to $O(k)$.

Definition 9.3 (i) Let M be a smooth manifold and $k \in \mathbb{N}$. For every $x \in M$, let $\mathcal{L}_k(M)|_x$ be the set of all linear maps $\mathbb{R}^k \to T|_x M$ and define

$$\mathcal{L}_k(M) := \bigcup_{x \in M} \mathcal{L}_k(M)|_x.$$

We also define $\pi_{\mathcal{L}_k(M)} : \mathcal{L}_k(M) \to M$ by $B \mapsto x$, if $B \in \mathcal{L}_k(M)|_x$.
(ii) Let (M, g) be a Riemannian manifold and $k \in \mathbb{N}$. Define a subset $O_k(M)$ of $\mathcal{L}_k(M)$ of all the elements $B \in \mathcal{L}_k(M)|_x$, $x \in M$, such that

$$\begin{aligned} &\text{if } 1 \leq k \leq \dim M, && \|Bu\|_g = \|u\|_{\mathbb{R}^k}\ ,\ \forall u \in \mathbb{R}^k \\ &\text{if } k \geq \dim M, && B \text{ surjective and } \|Bu\|_g = \|u\|_{\mathbb{R}^k}\ ,\ \forall u \in (\ker B)^\perp. \end{aligned}$$

Here $\|\cdot\|_{\mathbb{R}^k}$ is the euclidean norm in $\mathbb{R}^k$ and $\perp$ is taken with respect to the euclidean inner product in $\mathbb{R}^k$. Define also $\pi_{O_k(M)} := \pi_{\mathcal{L}_k(M)}|_{O_k(M)} : O_k(M) \to M$.

We will prove the following result.

Proposition 9.4 (i) For every $k \in \mathbb{N}$, the map $\pi_{\mathcal{L}_k(M)}$ is a smooth vector bundle over M, isomorphic to the direct sum bundle $\bigoplus_{i=1}^k TM \to M$.
(ii) If (M, g) is a Riemannian manifold of dimension n, then for all $k \in \mathbb{N}$, the map $\pi_{O_k(M)}$ defines a smooth sub-bundle of $\pi_{\mathcal{L}_k(M)}$ whose typical fiber is $O_k(\mathbb{R}^n)$.

Proof. (i) Define $\phi_k : \bigoplus_{i=1}^k TM \to \mathcal{L}_k(M)$ by

$$\phi_k(Y_1, \ldots, Y_k) := \sum_{i=1}^k \langle \cdot, e_i \rangle_{\mathbb{R}^k} Y_i.$$

Clearly, this is a bijection, linear in fibers and maps a fiber over a point x to a fiber over the same point. If we induce the differentiable structure on $\mathcal{L}_k(M)$ through ϕ_k, it is then trivial that $\pi_{\mathcal{L}_k(M)}$ becomes a smooth bundle isomorphic to $\bigoplus_{i=1}^k TM \to M$.

Before moving to prove the case (ii), we will build some natural local trivializations that will be used there. Given $x_0 \in M$, let $F = (X_1, \ldots, X_n)$ be a local frame of M

defined on some open neighbourhood U of x_0. Let θ_{X_i} be the 1-forms on U defined by setting $\theta_{X_i}(X_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Letting $n := \dim M$ and

$$t_F : (\pi_{\mathcal{L}_k(M)})^{-1}(U) \to U \times \mathcal{L}_k(\mathbb{R}^n); \quad B \mapsto \Big(x, u \mapsto \sum_{i=1}^n \theta_{X_i}|_x(Bu)e_i\Big), \quad \text{if } B \in \mathcal{L}_k(M)|_x,$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. Then

$$(t_F \circ \phi_k)(Y_1, \dots, Y_k) = \sum_{i=1}^n \sum_{j=1}^k \langle \cdot, e_j \rangle_{\mathbb{R}^k} \, \theta_{X_i|_x}(Y_j)e_i,$$

if $(Y_1, \dots, Y_k) \in \bigoplus_{i=1}^k T|_x M$, $x \in U$. Clearly $t_F \circ \phi_k$ is a smooth diffeomorphism $\bigoplus_{i=1}^k TU \to U \times \mathcal{L}_k(\mathbb{R}^n)$ and hence t_F is a smooth local trivialization of $\pi_{\mathcal{L}_k(M)}$.

(ii) Given $x_0 \in M$, let $F = (X_1, \dots, X_n)$ be a local g-orthonormal frame defined on some open neighbourhood U of x_0. Then it is easy to verify that $t_F(B) \in U \times O_k(\mathbb{R}^n)$ if and only if $B \in (\pi_{O_k(M)})^{-1}(U)$. Since $U \times O_k(\mathbb{R}^n)$ is a smooth (closed) submanifold of $U \times \mathcal{L}_k(\mathbb{R}^n)$ (by Proposition 9.2), we have that t_F^{-1} restricts to an embedding of $U \times O_k(\mathbb{R}^n)$ onto its image $(\pi_{O_k(M)})^{-1}(U)$, which is therefore a (closed) submanifold of $(\pi_{\mathcal{L}_k(M)})^{-1}(U)$. Thus $O_k(M)$ is a (closed) submanifold of $\mathcal{L}_k(M)$ and $\pi_{O_k(M)}$ is a smooth subbundle of $\pi_{\mathcal{L}_k(M)}$. □

Corollary 9.5 Let (M, g) be a connected Riemannian manifold of dimension n. If $k \in \mathbb{N}$ and $k \neq n$, then $O_k(M)$ is connected.

Proof. In this case, the base space M as well as the fiber $O_k(\mathbb{R}^n)$ of the bundle $\pi_{O_k(\mathbb{R}^n)}$ are connected (see Proposition 9.2), hence $O_k(M)$ is connected. □

Proposition 9.6 Let (M, g) be a connected Riemannian manifold of dimension n. If M is not orientable, then $O_n(M)$ is connected.

Proof. Write ∇ for the Levi-Civita connection of (M, g).

Let $x_0 \in M$. Recall first that by Propositions 9.2 and 9.4, the fiber $\pi_{O_n(M)}^{-1}(x_0)$ has two components, K_0 and K_1.

Since M is not orientable, there exists a loop $\omega \in \Omega_{x_0}(M)$, such that if $X = (X_1, \dots, X_n)$ is a given g-orthonormal frame at x_0, then $Y = (Y_1, \dots, Y_n)$ such that $Y_i = (P^\nabla)_0^1(\omega)X_i$, is a g-orthonormal frame which defines orientation opposite to that defined by X. Thus, if we define $B_0, B_1 \in O_n(M)|_{x_0}$ such that $B_0e_i := X_i$, $B_1e_i := Y_i$, it follows that B_0 and B_1 lie the fiber $\pi_{O_n(M)}^{-1}(x_0)$, say $B_0 \in K_0$, $B_1 \in K_1$.

Define $B(t) \in O_n(M)|_{\omega(t)}$ by setting $B(t)e_i := (P^\nabla)_0^t(\omega)X_i$ and notice that $B(t)$, $t \in [0, 1]$, is a path in $O_n(M)$ from B_0 to B_1.

Now, given $C_0 \in O_n(M)|_x$ for some $x \in M$, choose some piecewise smooth path $\gamma : [0, 1] \to M$, $\gamma(0) = x$, $\gamma(1) = x_0$. Defining $C(t) \in O_n(M)|_{\gamma(t)}$, $t \in [0, 1]$, by $C(t)e_i := (P^\nabla)_0^t(\gamma)(C_0e_i)$, we have a path $C(\cdot)$ in $O_n(M)$ from $C_0 = C(0)$ to $C(1) \in O_n(M)|_{x_0}$.

But then $C(1) \in K_0$ or $C(1) \in K_1$. In the former case, we may use the connectedness of K_0 to join $C(1)$ continuously to B_0 inside K_0 and therefore joining this path to $C(\cdot)$, we have a continuous path from C_0 to B_0. On the other hand, if $C(1) \in K_1$, we can choose a path in K_1 that joins $C(1)$ to B_1 and then traverse the path $B(\cdot)$ backwards from B_1 to B_0. Joining these path we obtain, again, a continuous curve in $O_n(M)$ from C_0 to B_0, therefore finishing the proof of the proposition. $\square$

10 Rolling of Affine Manifolds

In this section, we assume that (M, ∇) and $(\hat{M}, \hat{\nabla})$ are affine connected manifolds of dimensions n and $\hat{n}$, respectively. The definition of the state space $Q(M, \hat{M})$ that appear in Lemma 8.3 does not in general make sense in this setting for two reasons: there might not be Riemannian metrics that are compatible w.r.t $\nabla, \hat{\nabla}$ and the dimensions $n, \hat{n}$ might be different. If it were the case that $n = \hat{n}$, then a possible replacement of $Q(M, \hat{M})$ might be $GL(TM, T\hat{M}) := \{A \in T^*M \otimes T\hat{M} \mid \det(A) \neq 0\}$. Since generally $n \neq \hat{n}$, we should expect that our only objective choice for the substitute of $Q(M, \hat{M})$ should be $T^*M \otimes T\hat{M}$. We record this in a definition.

Definition 10.1 We call $T^*M \otimes T\hat{M}$ *generalized state space* for rolling of an affine manifold (M, ∇) against $(\hat{M}, \hat{\nabla})$.

Let us show that $T^*M \otimes T\hat{M}$ is a bundle over M and $\hat{M}$.

Proposition 10.2 For any given any manifolds $M, \hat{M}$,
(i) the map $\pi_{T^*M\otimes T\hat{M},M} : T^*M \otimes T\hat{M} \to M$ defines a bundle whose typical fiber is diffeomorphic to $\bigoplus_{i=1}^{n} T\hat{M}$;
(ii) the map $\pi_{T^*M\otimes T\hat{M},\hat{M}} : T^*M \otimes T\hat{M} \to \hat{M}$ defines a bundle whose typical fiber is diffeomorphic to $\bigoplus_{i=1}^{\hat{n}} T^*M$.

Proof. (i) Given $x_0 \in M$, choose any frame $F = (X_1, \dots, X_n)$, with $n = \dim M$, defined on some neighbourhood U of x_0, set

$$\tau_F : (\pi_{T^*M\otimes T\hat{M},M})^{-1}(U) \to U \times \bigoplus_{i=1}^{n} T\hat{M}; \quad \tau_F(x, \hat{x}; A) := \big(x, (AX_1|_x, \dots, AX_n|_x)\big).$$

These maps can be taken as local trivializations of $\pi_{T^*M\otimes T\hat{M},M}$. Indeed, each τ_F is a diffeomorphism, which is seen by writing explicitly its inverse map: Writing θ_F^i for the element of $\Lambda^1(U)$ defined by requiring that $\theta_F^i(X_j) = \delta_j^i$, with δ_j^i the Kronecker symbol, we have

$$\tau_F^{-1} : U \times \bigoplus_{i=1}^{n} T\hat{M} \to (\pi_{T^*M\otimes T\hat{M},M})^{-1}(U);$$
$$\tau_F^{-1}\big(x, (\hat{X}_1, \dots, \hat{X}_n)\big) = \Big(x, \hat{x}; \sum_{i=1}^{n} \theta_F^i|_x \otimes \hat{X}_i\Big), \quad \text{if } (\hat{X}_1, \dots, \hat{X}_n) \in \bigoplus_{i=1}^{n} T|_{\hat{x}}\hat{M}.$$

Here the expression $\sum_{i=1}^{n} \theta_F^i|_x \otimes \hat{X}_i$ stands for the linear map $T|_x M \to T|_{\hat{x}}\hat{M}$; $Y \mapsto \sum_{i=1}^{n} \theta_F^i|_x(Y)\hat{X}_i$.

(ii) Given $\hat{x}_0 \in \hat{M}$, choose any co-frame $\hat{\Theta} = (\hat{\theta}_1, \dots, \hat{\theta}_{\hat{n}})$, with $\hat{n} = \dim \hat{M}$, defined on some neighbourhood $\hat{U}$ of $\hat{x}_0$, set

$$\hat{\tau}_{\hat{\Theta}} : (\pi_{T^*M\otimes T\hat{M},\hat{M}})^{-1}(U) \to \hat{U} \times \bigoplus_{i=1}^{\hat{n}} T^*M; \quad \hat{\tau}_{\hat{\Theta}}(x, \hat{x}; A) := \big(\hat{x}, (A^*\hat{\theta}_1|_{\hat{x}}, \dots, A^*\hat{\theta}_{\hat{n}}|_{\hat{x}})\big),$$

where $A^* : T^*|_{\hat{x}}\hat{M} \to T^*|_x M$ is the dual map of $A : T|_x M \to T|_{\hat{x}}\hat{M}$. The map $\hat{\tau}_{\hat{\Theta}}$ is obviously smooth and its smooth inverse map can be build in the analogous way as in (i). This completes the proof. □

One can readily take, in this setting, the rolling dynamics to be that described in Proposition 8.6.

Definition 10.3 One says that an a.c. curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [a, b]$, in $T^*M \otimes T\hat{M}$ describes *rolling of* (M, ∇) *against* $(\hat{M}, \hat{\nabla})$ if for almost every $t \in [a, b]$ the following conditions hold:
(i) No-slipping: $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$,
(ii) No-spinning: $A(t)\nabla_{\dot{\gamma}(t)}X(t) = \hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(t)X(t))$ for all vector fields X along γ.
We call a curve $q(t)$ satisfying the above conditions a *rolling curve* in $T^*M \otimes T\hat{M}$.

The next easy lemma gives a characterization the rolling curves, which will be useful for the definition of a distribution in $T^*M \otimes T\hat{M}$ that controls the rolling curves.

Lemma 10.4 An absolutely continuous curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [a, b]$, in $T^*M \otimes T\hat{M}$ is a rolling curve if and only if for a.e. $t \in [a, b]$,
(i) $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$,
(ii) $A(t) = (P^{\hat{\nabla}})^t_a(\hat{\gamma}) \circ A(a) \circ (P^{\nabla})^a_t(\gamma)$.

Proof. First, suppose $q(t)$ is a rolling curve. Take arbitrary $X_0 \in T|_{\gamma(a)}M$ and define $X(t) := (P^{\nabla})^t_a(\gamma)X_0$. Then $\hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(t)X(t)) = A(t)\nabla_{\dot{\gamma}(t)}X(t) = 0$ i.e. $A(t)X(t) = (P^{\hat{\nabla}})^t_a(\hat{\gamma})(A(a)X_0)$ i.e. $(A(t) \circ (P^{\nabla})^t_a(\gamma))X_0 = (P^{\hat{\nabla}})^t_a(\hat{\gamma}) \circ A(a))X_0$. Thus (ii) follows since X_0 was arbitrary.

Conversely, assume that $q(t)$ satisfies (i) and (ii) of the statement of this lemma. Given a vector field $X(t)$ along γ, let $X^0_1, \dots, X^0_n$ be a basis of $T|_{\gamma(a)}M$, $X_i(t) := (P^{\nabla})^t_a(\gamma)X^0_i$ and let $a_i : [a, b] \to \mathbb{R}$ be the absolutely continuous functions such that $X(t) = \sum_i a_i(t)X_i(t)$. Then for all t, $A(t)X(t) = \sum_i a_i(t)(P^{\hat{\nabla}})^t_a(\hat{\gamma})(A(a)X^0_i)$ and thus for a.e. t,

$$\hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(t)X(t)) = \sum_i \dot{a}_i(t)(P^{\hat{\nabla}})^t_a(\hat{\gamma})(A(a)X^0_i) = A(t)\big(\sum_i \dot{a}_i(t)X_i(t)\big).$$

Since $\sum_i \dot{a}_i(t)X_i(t) = \nabla_{\dot{\gamma}(t)}X(t)$, we have $\hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(t)X(t)) = A(t)\nabla_{\dot{\gamma}(t)}X(t)$. This completes the proof.

□

We omit the straightforward proof of the next proposition, which gives some basic properties of $\mathscr{L}_{\mathrm{R}}$ and $\mathcal{D}_{\mathrm{R}}$ (to be compared with Lemma 3.6 and Theorem 4.25 in paper D). In particular, it provides the justification for the Definition 3.5.

Proposition 10.5 (i) The map $(\pi_{T^*M\otimes T\hat{M},M})_*$ restricted to $\mathcal{D}_{\mathrm{R}}|_q$ gives an isomorphism $\mathcal{D}_{\mathrm{R}}|_q \to T|_x M$, its inverse map being $X \mapsto \mathscr{L}_{\mathrm{R}}(X)|_q$. Consequently, $\mathcal{D}_{\mathrm{R}}$ has constant rank $n = \dim M$.

(ii) If $X \in \mathrm{VF}(M)$, then the vector field $\mathscr{L}_{\mathrm{R}}(X)$ on $T^*M \otimes T\hat{M}$ defined by $q = (x, \hat{x}; A) \mapsto \mathscr{L}_{\mathrm{R}}(X|_x)|_q$ is smooth. Consequently, the rolling distribution $\mathcal{D}_{\mathrm{R}}$ is smooth.

(iii) An a.c. curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [a, b]$, in $T^*M \otimes T\hat{M}$ is a rolling curve if and only if it is tangent to $\mathcal{D}_{\mathrm{R}}$.

Next we prove a uniqueness type result, which justifies Definition 3.7.

Lemma 10.6 Let $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ and $q_1(t) = (\gamma(t), \hat{\gamma}_1(t); A_1(t))$, $t \in [a, b]$, be rolling curves such that $q_1(a) = q(a)$, then $q_1(t) = q(t)$ for all $t \in [a, b]$. Thus a rolling curve q is uniquely determined by the projected curve γ in M and the initial condition $q(a)$ in $T^*M \otimes T\hat{M}$.

Proof. To simplify a little the argument, which is completely standard, we assume that q, q_1 are smooth and that γ is an embedding into M. Thus we may find a smooth vector field X in M such that γ is its integral curve through $\gamma(a)$. Since $X|_{\gamma(t)} = \dot{\gamma}(t) = (\pi_{T^*M\otimes T\hat{M},M})_*\dot{q}(t)$ and $\dot{q}(t) \in \mathcal{D}_{\mathrm{R}}|_{q(t)}$, it follows that $\dot{q}(t) = \mathscr{L}_{\mathrm{R}}(X|_{\gamma(t)})|_{q(t)}$. Similarly, $\dot{q}_1(t) = \mathscr{L}_{\mathrm{R}}(X|_{\gamma(t)})|_{q_1(t)}$ and thus q and q_1 are integral curves of the smooth vector field $\mathscr{L}_{\mathrm{R}}(X)$ on $T^*M \otimes T\hat{M}$ through $q(a) = q_1(a)$. By uniqueness of integral curves, $q(t) = q_1(t)$ for all $t \in [a, b]$.

□

From now on, we will assume that curves are defined *a priori* on $[0, 1]$ i.e. we take $a = 0$, $b = 1$. This creates no loss of generality, except when the existence of $\mathcal{D}_{\mathrm{R}}$-lifts is concerned (see the next proposition). Basic properties of $\mathcal{D}_{\mathrm{R}}$-lifts and orbits of $\mathcal{D}_{\mathrm{R}}$ are collected in the two propositions that follows.

Proposition 10.7 If $q_0 = (x_0, \hat{x}_0; A_0) \in T^*M \otimes T\hat{M}$ and $\gamma(t)$, $t \in [0, 1]$ is an a.c. curve in M with $\gamma(0) = x_0$, then there exists a T, $0 < T \leq 1$ and a unique $\mathcal{D}_{\mathrm{R}}$-lift $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)$ of γ through q_0, defined for $t \in [0, T]$. In other words, the $\mathcal{D}_{\mathrm{R}}$-lift of $\gamma|_{[0,T]}$ through q_0 exists.

Moreover, if $(\hat{M}, \hat{\nabla})$ is geodesically complete, and if γ is a piecewise geodesic, then one can take above $T = 1$.

The proof will make use of the following lemma.

Lemma 10.8 On an affine manifold (M, ∇), given an a.c. continuous curve $\gamma : [0, 1] \to M$, we define its ∇-*development* on $T|_{\gamma(0)}M$ to be the a.c. curve,

$$\Lambda^{\nabla}_{\gamma(0)}(\gamma) : [0, 1] \to T|_{\gamma(0)}M; \quad \Lambda^{\nabla}_{\gamma(0)}(\gamma)(t) := \int_0^t (P^{\nabla})^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s.$$

Then, given $x_0 \in M$, for any a.c. curve $\Gamma : [0, 1] \to T|_{x_0}M$, there exists a T, $0 < T \leq 1$, and a unique a.c. curve $\gamma : [0, T] \to M$ such that $\gamma(0) = x_0$ and $\Lambda^{\nabla}_{x_0}(\gamma)(t) = \Gamma(t)$ for all

$t \in [0, T]$.

If, moreover, (M, ∇) is geodesically complete and if Γ is piecewise linear, then one can take above $T = 1$.

Proof. See [15], Chapter III. $\square$

Remark 10.9 If $\gamma : [0, T] \to M$ and $\Gamma : [0, T] \to T|_{x_0}M$ are such that $\Lambda^{\nabla}_{x_0}(\gamma)(t) = \Gamma(t)$, $t \in [0, T]$, then it is clear that γ is a piecewise geodesic if and only if Γ is a piecewise linear curve.

Here is the proof of Proposition 10.7.

Proof. Since $A_0 \circ \Lambda^{\nabla}_{x_0}(\gamma)(t)$, $t \in [0, 1]$, is an absolutely continuous curve in $T|_{\hat{x}_0}\hat{M}$, there exists, by the preceding lemma, a T, $0 < T \leq 1$, and a unique a.c. curve $\hat{\gamma} : [0, T] \to \hat{M}$, $\hat{\gamma}(0) = \hat{x}_0$, such that $\Lambda^{\hat{\nabla}}_{\hat{x}_0}(\hat{\gamma})(t) = A_0 \Lambda^{\nabla}_{x_0}(\gamma)(t)$. We claim that a curve in Q, $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, where

$$A(t) = (P^{\hat{\nabla}})^t_0(\hat{\gamma}) \circ A_0 \circ (P^{\nabla})^0_t(\gamma), \quad t \in [0, T],$$

Indeed, by Lemma 10.4, it suffices to show that $\dot{\hat{\gamma}}(t) = A(t)\dot{\gamma}(t)$, for all $t \in [0, T]$. But this follows easily from the definition of development and that of $\hat{\gamma}(t)$ and $A(t)$,

$$(P^{\hat{\nabla}})^0_t(\hat{\gamma})\dot{\hat{\gamma}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{\hat{\nabla}}_{\hat{x}_0}(\hat{\gamma})(t) = A_0 \frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{\nabla}_{x_0}(\gamma)(t) = \big(A_0 \circ (P^{\nabla})^0_t(\gamma)\big)\dot{\gamma}(t).$$

Assuming then that $(\hat{M}, \hat{\nabla})$ is geodesically complete, if γ is a piecewise geodesic on M, then $\Lambda^{\nabla}_{x_0}(\gamma)$ is a piecewise linear curve and hence so is $A_0 \circ \Lambda^{\nabla}_{x_0}(\gamma)$. By Lemma 10.8 then, we may take for the domain of definition of $\hat{\gamma}$ the whole interval $[0, 1]$ i.e. $T = 1$. $\square$

Remark 10.10 An alternative argument for the proof of the previous proposition (and to Lemma 10.8) can be given by following the idea in the proof of Lemma 10.6 above. Indeed, assuming that $\gamma : [0, T_0] \to M$ is a smooth embedding for some T_0, $0 < T_0 \leq 1$, choose a neighbourhood U of the image of γ in M and a smooth vector field $X \in \mathrm{VF}(M)$ such that $X|_{\gamma(t)} = \dot{\gamma}(t)$ for $t \in [0, T_0]$. Then let $q(t)$ be an integral curve of the vector field $\mathscr{L}_{\mathrm{R}}(X)$, defined on $\pi^{-1}_{Q,M}(U)$, starting from q_0. This integral curve $q(t)$ is then defined on some interval $t \in [0, T_1]$, $T_1 > 0$. Taking $T = \min\{T_0, T_1\}$, we have that $q(t)$, $t \in [0, T]$, is the sought rolling curve $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)$, $t \in [0, T]$.

Remark 10.11 As we will mention in the next section, in Riemannian case the completeness of $(\hat{M}, \hat{\nabla})$, i.e. completeness of $(\hat{M}, \hat{g})$, implies that, in the above proposition, one can take $T = 1$ for *any* absolutely continuous γ, not just for piecewise geodesic ones.

Proposition 10.12 If $(\hat{M}, \hat{\nabla})$ is geodesically complete, then for every $q_0 \in T^*M \otimes T\hat{M}$ the map $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to M$; $(x, \hat{x}; A) \mapsto x$ is a bundle map.

Proof. Fix $q_0 = (x_0, \hat{x}_0; A_0) \in T^*M \otimes T\hat{M}$.

First we make the observation that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to M$ is surjective. Indeed, if $x \in M$, let $\gamma(t)$, $t \in [0,1]$, be a piecewise geodesic such that $\gamma(0) = x_0$, $\gamma(1) = x$. Then by (i), $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1)$ exists and since it belongs to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, we have that $x = \pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1))$ i.e. x belongs to the image of $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$.

Next we observe that since the map $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to M$ is a smooth (surjective) submersion, each fiber $(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x)$, $x \in M$, is a closed submanifold of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$.

Finally, given $x_1 \in M$, let U be a starlike neighbourhood of 0 in $T|_{x_1}M$ such that $\exp^{\nabla}_{x_1}|_U : U \to M$ is a diffeomorphism onto its image V. Writing

$$\gamma_x(t) := \exp^{\nabla}_{x_1}\big(t(\exp^{\nabla}_{x_1}|_U)^{-1}(x)\big),\ \forall x \in V,\ t \in [0,1],$$

we define

$$\tau_U : (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(V) \to V \times (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x_1); \quad q = (x, \hat{x}; A) \mapsto \big(x, q_{\mathcal{D}_{\mathrm{R}}}(\gamma_x^{-1}, q)(1)\big).$$

This is obviously a well defined smooth map, and because its inverse map is given by

$$\tau_U^{-1} : V \times (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x_1) \to (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(V); \quad (x, q_1) \mapsto q_{\mathcal{D}_{\mathrm{R}}}(\gamma_x, q_1)(1),$$

we conclude that τ_U is a diffeomorphism, hence a local trivialization of $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ around x_1.

□

Next lemma implies that the system $(T^*M \otimes T\hat{M}, \mathcal{D}_{\mathrm{R}})$ can never be completely controllable.

Proposition 10.13 Define for $k = 0, 1, \ldots, \min\{n, \hat{n}\}$,

$$r_k(M, \hat{M}) := \{(x, \hat{x}; A) \in T^*M \otimes T\hat{M} \mid A \text{ has rank } k\}.$$

Then

$$\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \subset r_k(M, \hat{M}), \quad \forall q \in r_k(M, \hat{M}).$$

Proof. Let $q_0 = (x_0, \hat{x}_0; A_0) \in r_k(M, \hat{M})$ and $q_1 \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$. Then by Proposition 10.5 case (iii) there exists an a.c. rolling curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in $T^*M \otimes T\hat{M}$ such that $q(0) = q_0$, $q(1) = q_1$. But by Lemma 10.4 case (ii) then, $A(t) = (P^{\hat{\nabla}})^t_0(\hat{\gamma}) \circ A_0 \circ (P^{\nabla})^0_t(\gamma)$. But the fact that $(P^{\hat{\nabla}})^t_0(\hat{\gamma}) : T|_{\hat{x}_0}\hat{M} \to T|_{\hat{\gamma}(t)}\hat{M}$ and $(P^{\nabla})^0_t(\gamma) : T|_{\gamma(t)}M \to T|_{x_0}\hat{M}$ are invertible mappings, implies that rank $A(t) =$ rank $A_0 = k$ and hence in particular $A_1 = A(1)$ has rank k, i.e. $q_1 \in r_k(M, \hat{M})$. □

Remark 10.14 The above lemma implies that controllability issues restrict to spaces $r_k(M, \hat{M})$. If spaces (M, ∇), $(\hat{M}, \hat{\nabla})$ have some extra structure, then one might be able to *reduce* the study to subspaces of $r_k(M, \hat{M})$. In particular, this is the case (M, ∇), $(\hat{M}, \hat{\nabla})$ are Riemannian spaces with Levi-Civita connections, subject which is the topic of the next section.

11 Rolling of Riemannian Manifolds

In this section, we assume that (M, g) and $(\hat{M}, \hat{g})$ are connected Riemannian manifolds of dimensions n and $\hat{n}$, respectively. Also, we will be writing ∇ and $\hat{\nabla}$ for the respective Levi-Civita connections.

As we have already remarked, the definition of $Q(M, \hat{M})$ in Lemma 8.3 would be valid in this higher dimensional (abstract) setting if $n = \hat{n}$ and if $M, \hat{M}$ were oriented. One can actually generalize slightly this definition to deal also with the cases $n \neq \hat{n}$ as well as the non-oriented cases.

Definition 11.1 Let (M, g), $(\hat{M}, \hat{g})$ be connected Riemannian manifolds of dimensions n, $\hat{n}$. Then the *state space* $Q(M, \hat{M})$ *for rolling of* (M, g) *against* $(\hat{M}, \hat{g})$ is defined as follows:

(i) If $n \leq \hat{n}$, then

$$Q(M, \hat{M}) := \{(x, \hat{x}; A) \in T^*M \otimes T\hat{M} \mid \|AX\|_{\hat{g}} = \|X\|_g,\ \forall X \in T|_x M\};$$

(ii) If $n \geq \hat{n}$, then

$$Q(M, \hat{M}) := \{(x, \hat{x}; A) \in T^*M \otimes T\hat{M} \mid \|AX\|_{\hat{g}} = \|X\|_g,\ \forall X \in (\ker A)^{\perp}$$
$$A \text{ surjective}\}.$$

Moreover, if $n = \hat{n}$ and if $M, \hat{M}$ are both oriented, we define

$$Q_+(M, \hat{M}) := \{(x, \hat{x}; A) \in Q(M, \hat{M}) \mid \det A > 0\}.$$

Define also

$$\pi_{Q(M,\hat{M}),M} := \pi_{T^*M\otimes T\hat{M},M}|_{Q(M,\hat{M})} : Q(M, \hat{M}) \to M$$
$$\pi_{Q(M,\hat{M}),\hat{M}} := \pi_{T^*M\otimes T\hat{M},\hat{M}}|_{Q(M,\hat{M})} : Q(M, \hat{M}) \to \hat{M}.$$

Remark 11.2 (1) Notice that the definitions (i) and (ii) of $Q(M, \hat{M})$ above coincide in the case where $n = \hat{n}$.

(2) Even though the notation $Q((M, g), (\hat{M}, \hat{g}))$ would be more appropriate above, we rather avoid this more cumbersome notation and write $Q(M, \hat{M})$, and usually even just Q, for the state space, when the spaces (M, g), $(\hat{M}, \hat{g})$ are clear from the context.

(3) In Parts I and IV as well as in Papers A-B we used Q or $Q(M, \hat{M})$ to denote $Q_+(M, \hat{M})$.

Some very basic properties of $Q(M, \hat{M})$ are collected in the next proposition.

Proposition 11.3 (i) The space $Q(M, \hat{M})$ is a smooth closed submanifold of $T^*M \otimes T\hat{M}$ of dimension

$$\dim Q(M, \hat{M}) = n + \hat{n} + n\hat{n} - \frac{N(N+1)}{2}, \quad \text{where } N := \min\{n, \hat{n}\},$$

and $\pi_{Q(M,\hat{M}),M}$ is a smooth subbundle of $\pi_{T^*M\otimes T\hat{M},M}$ with typical fiber $O_n(\hat{M})$.

(ii) For $A \in T^*|_x M \otimes T|_{\hat{x}} \hat{M}$, let $A^T \in T^*|_{\hat{x}} \hat{M} \otimes T|_x M$ denote the transpose of A with respect to the inner products $g, \hat{g}$. Then the map

$$\tau : T^*M \otimes T\hat{M} \to T^*\hat{M} \otimes TM; \quad (x, \hat{x}; A) \mapsto (\hat{x}, x; A^T)$$

is a diffeomorphism and its restriction to $Q(M, \hat{M})$ gives a diffeomorphism

$$\tau : Q(M, \hat{M}) \to Q(\hat{M}, M).$$

(iii) If $n \neq \hat{n}$ or if one of $M, \hat{M}$ is not orientable, then the space $Q(M, \hat{M})$ is connected. Otherwise, i.e. when $n = \hat{n}$ and $M, \hat{M}$ orientable, $Q(M, \hat{M})$ has two components, one of which is $Q_+(M, \hat{M})$.

Proof. (i) Let $x_0 \in M$ and choose any local g-orthonormal frame $(X_1, \dots, X_n)$ defined on some neighbourhood U of x_0 and let τ_F be the local trivialization of $\pi_{T^*M\otimes T\hat{M},M}$ as given in the proof of Proposition 10.2. Then, if we let $\hat{\phi}_k : \bigoplus_{i=1}^k T\hat{M} \to \mathcal{L}_k(\hat{M})$ be the bundle isomorphism as defined in the proof of Proposition 9.4, the map $\overline{\tau}_F := (\mathrm{id}_U \times \hat{\phi}_n) \circ \tau_F$ is also a local trivialization of $\pi_{T^*M\otimes T\hat{M},M}$ around x_0.

Indeed, let $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ be such that $x \in U$. Then $\overline{\tau}_F(q) = \big(x, \sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x\big)$, and so if $u, v \in \mathbb{R}^n$, we have

$$\hat{g}\Big(\sum_{i=1}^n \langle u, e_i \rangle AX_i|_x, \sum_{j=1}^n \langle v, e_i \rangle AX_j|_x\Big) = \sum_{i,j=1}^n \langle u, e_i \rangle \langle v, e_j \rangle \hat{g}(AX_i|_x, AX_j|_x).$$

Consider first the case $n \leq \hat{n}$. If $q \in Q(M, \hat{M})$, then $\hat{g}(AX_i|_x, AX_j|_x) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, and hence $\|\sum_{i=1}^n \langle u, e_i \rangle AX_i|_x\|_{\hat{g}}^2 = \sum_{i=1}^n \langle u, e_i \rangle^2 = \|u\|_{\mathbb{R}^n}^2$, i.e. $\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x \in O_n(\hat{M})$. Conversely, if $\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x \in O_n(\hat{M})$, then $\sum_{i,j=1}^n \langle u, e_i \rangle \langle v, e_j \rangle \hat{g}(AX_i|_x, AX_j|_x) = \langle u, v \rangle$ for all $u, v \in \mathbb{R}^n$. Taking $u = e_i$, $v = e_j$, we thus get $\hat{g}(AX_i|_x, AX_j|_x) = \delta_{ij}$, which means that $q = (x, \hat{x}; A) \in Q(M, \hat{M})$.

Let us suppose then that $n \geq \hat{n}$. First, it is clear that A is surjective $T|_x M \to T|_{\hat{x}} \hat{M}$ if and only if $\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x$ is surjective $\mathbb{R}^n \to T|_{\hat{x}} \hat{M}$. Then, notice that $X = \sum_{i=1}^n u_i X_i|_x \in T|_x M$ belongs to $\ker A$ if and only if $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ belongs to $\ker(\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x)$ and hence that X belongs to $(\ker A)^\perp$ if and only if the corresponding u belongs to $\ker(\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x)^\perp$. Therefore, if $q \in Q(M, \hat{M})$ and $u \in \ker(\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x)^\perp$, then $X := \sum_{i=1}^n u_i X_i|_x \in (\ker A)^\perp$, and we have $\|\sum_{i=1}^n \langle u, e_i \rangle AX_i|_x\|_{\hat{g}} = \|AX\|_{\hat{g}} = \|X\|_g$, thus $\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x \in O_n(\hat{M})$. Conversely, if $\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x \in O_n(\hat{M})$, then for $X = \sum_{i=1}^n u_i X_i|_x \in (\ker A)^\perp$, we have that $u = (u_1, \dots, u_n) \in \ker(\sum_{i=1}^n \langle \cdot, e_i \rangle AX_i|_x)^\perp$ and so $\|AX\|_{\hat{g}} = \|\sum_{i=1}^n \langle u, e_i \rangle AX_i|_x\|_{\hat{g}} = \|u\|_{\mathbb{R}^m} = \|X\|_g$. This proves our claim.

Hence, the diffeomorphism $\overline{\tau}_F^{-1} : U \times \mathcal{L}_n(\hat{M}) \to (\pi_{T^*M\otimes T\hat{M},\hat{M}})^{-1}(U)$ restricts to an embedding of the closed submanifold $U \times O_n(\hat{M})$ of $U \times \mathcal{L}_n(\hat{M})$ onto its image $Q(M, \hat{M}) \cap (\pi_{T^*M\otimes T\hat{M},\hat{M}})^{-1}(U) = (\pi_{Q(M,\hat{M}),M})^{-1}(U)$, which therefore is a closed submanifold of $(\pi_{T^*M\otimes T\hat{M},\hat{M}})^{-1}(U)$. This proves that $Q(M, \hat{M})$ is a closed submanifold of $T^*M \otimes T\hat{M}$ and that $\pi_{Q(M,\hat{M}),M}$ is a smooth sub-bundle of $\pi_{T^*M\otimes T\hat{M},M}$, with local trivializations the restrictions of $\overline{\tau}_F$ onto $(\pi_{Q(M,\hat{M}),M})^{-1}(U)$ and with typical fiber $O_n(\hat{M})$. Finally this implies that

$$\dim Q(M,\hat{M}) = n + \dim O_n(\hat{M}) = n + \hat{n} + \dim O_n(\mathbb{R}^{\hat{n}}) = n + \hat{n} + n\hat{n} - \frac{N(N+1)}{2},$$

where $N = \min\{n, \hat{n}\}$. This proves (i).

(ii) Writing more explicitly $\tau_{M,\hat{M}}$ for the map $T^*M \otimes T\hat{M}$ such that $(x,\hat{x};A) \to (\hat{x},x;A^T)$, we see that $\tau_{\hat{M},M}$ is the inverse of the map $\tau_{M,\hat{M}}$, thus $\tau_{M,\hat{M}}$ is a diffeomorphism. Moreover, for $q = (x,\hat{x};A) \in T^*M \otimes T\hat{M}$, it is clear that $q \in Q(M,\hat{M})$ if and only if $\tau(q) \in Q(\hat{M},M)$. This gives (ii).

(iii) If $n \neq \hat{n}$, then by Corollary 9.5, we have that the typical fiber $O_n(\hat{M})$ of $\pi_{Q(M,\hat{M}),M}$ is connected (recall that $\hat{M}$ is assumed to be connected). Since the base space M is also connected, we have that $Q(M,\hat{M})$ is connected.

Assume that $n = \hat{n}$. If $\hat{M}$ is not orientable, we have that $O_n(\hat{M})$ is connected by Proposition 9.6, and hence, again, so is $Q(M,\hat{M})$. Using (ii) to change the roles of M and $\hat{M}$, we conclude that $Q(M,\hat{M})$ is connected also, if M is not orientable.

So it remains to prove that,if M and $\hat{M}$ are orientable and $n = \hat{n}$, then $Q(M,\hat{M})$ has two components, one of which is $Q_+(M,\hat{M}) = \{q = (x,\hat{x};A) \in Q(M,\hat{M}) \mid \det A > 0\}$, while the other is $Q_-(M,\hat{M}) := \{q = (x,\hat{x};A) \in Q(M,\hat{M}) \mid \det A < 0\}$ (which make sense once orientations have been chosen).

Clearly both $Q_+(M,\hat{M})$ and $Q_-(M,\hat{M})$ are disjoint, are both open and closed in $Q(M,\hat{M})$ and their union is $Q(M,\hat{M})$. Hence it is enough to show that they are connected. We do this only for $Q_+(M,\hat{M})$ since the argument is the same for $Q_-(M,\hat{M})$.

Write $\pi_{Q_+} : Q_+(M,\hat{M}) \to M \times \hat{M};\ (x,\hat{x};A) \mapsto (x,\hat{x})$. Fix $(x_0,\hat{x}_0) \in M \times \hat{M}$ and choose some local oriented orthonormal frames $F = (X_1,\dots,X_n)$, $\hat{F} = (\hat{X}_1,\dots,\hat{X}_n)$ defined on some open neighbourhoods U, $\hat{U}$ of x_0, $\hat{x}_0$, respectively. For $q = (x,\hat{x};A) \in Q(M,\hat{M})$, let $G_{F,\hat{F}}(A)$ be the $n\times n$-matrix whose i-th row and j-th column is $\hat{g}(\hat{X}_i, AX_j)$. Clearly $G_{F,\hat{F}}(A) \in \mathrm{SO}(n)$ if and only if $q \in Q_+(M,\hat{M})$. Then the map

$$\tau_{F,\hat{F}} : (\pi_{Q_+})^{-1}(U \times \hat{U}) \to (U \times \hat{U}) \times \mathrm{SO}(n);\quad (x,\hat{x};A) \mapsto ((x,\hat{x}), G_{F,\hat{F}}(A))$$

is seen without difficulty to be a diffeomorphism, which proves that π_{Q_+} is a smooth bundle, whose base space $M \times \hat{M}$ is connected and whose fiber $\mathrm{SO}(n)$ is connected, hence the total space $Q_+(M,\hat{M})$ is connected. This finishes the proof of (iii). □

Thus for given (M,g), $(\hat{M},\hat{g})$, the state spaces (resp. generalized state spaces) $Q(M,\hat{M})$ and $Q(\hat{M},M)$ (resp. $T^*M \otimes T\hat{M}$ and $T^*\hat{M} \otimes TM$) are identical up to diffeomorphism. We can now ask whether the distribution $\mathcal{D}_{\mathrm{R}}$ (and hence the rolling model) nicely restricts from $T^*M \otimes T\hat{M}$ to $Q(M,\hat{M})$. The answer, as given in next result, is positive but it is no longer true that the control systems in $Q(M,\hat{M})$ and in $Q(\hat{M},M)$ are equivalent, except when $n = \hat{n}$.

Proposition 11.4 (i) For all $q \in Q(M,\hat{M})$ one has $\mathcal{D}_{\mathrm{R}}|_q \subset T|_q Q(M,\hat{M})$. Thus $\mathcal{D}_{\mathrm{R}}$ restricts to a smooth distribution of rank n on $Q(M,\hat{M})$. We still write this restriction as $\mathcal{D}_{\mathrm{R}}$ and call it the *rolling distribution* for rolling of (M,g) against $(\hat{M},\hat{g})$.

(ii) Assume that $n \leq \hat{n}$ and write $\widehat{\mathcal{D}_{\mathrm{R}}}$ for the rolling distribution in $Q(\hat{M}, M)$ (i.e. for rolling of $(\hat{M}, \hat{g})$ against (M, g)). Then if $\tau : Q(M, \hat{M}) \to Q(\hat{M}, M)$ is as in Proposition 11.3 case (ii), we have

$$\tau_* \mathcal{D}_{\mathrm{R}} \subset \widehat{\mathcal{D}_{\mathrm{R}}}$$

and therefore, in particular, $\tau(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)) \subset \mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\tau(q))$ for all $q \in Q(M, \hat{M})$.

Proof. (i) It is enough to show that $q(t)$, $t \in [0,1]$, is any rolling curve such that $q(0) \in Q(M, \hat{M})$, then $q(t) \in Q(M, \hat{M})$ for all $t \in [0,1]$. Indeed, if $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, then $A(t) = (P^{\hat{\nabla}})_0^t(\hat{\gamma}) \circ A(0) \circ (P^{\nabla})_t^0(\gamma)$. But since $(P^{\nabla})_t^0(\gamma) : T|_{\gamma(t)} M \to T|_{\gamma(0)} M$ is a g-isometry, $(P^{\hat{\nabla}})_0^t(\hat{\gamma}) : T|_{\hat{\gamma}(0)} \hat{M} \to T|_{\hat{\gamma}(t)} \hat{M}$ is a $\hat{g}$-isometry and $A(0) \in Q(M, \hat{M})$, it follows that if $n \leq \hat{n}$, then for all $X \in T|_x M$,

$$\|A(t)X\|_{\hat{g}} = \left\|\big(A(0) \circ (P^{\nabla})_t^0(\gamma)\big)X\right\|_{\hat{g}} = \left\|(P^{\nabla})_t^0(\gamma)X\right\|_g = \|X\|_g ,$$

while if $n \geq \hat{n}$, then since $A(0)$ is surjective, $A(t)$ is surjective for all $t \in [0,1]$, and for any $X \in (\ker A(t))^{\perp}$, one has that $(P^{\nabla})_t^0(\gamma)X \in (\ker A(0))^{\perp}$ and thus

$$\|A(t)X\|_{\hat{g}} = \left\|\big(A(0) \circ (P^{\nabla})_t^0(\gamma)\big)X\right\|_{\hat{g}} = \left\|(P^{\nabla})_t^0(\gamma)X\right\|_g = \|X\|_g .$$

This shows that $q(t) \in Q(M, \hat{M})$ for all $t \in [0,1]$.

(ii) Let $q_0 \in Q(M, \hat{M})$ and let $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [0,1]$, be an arbitrary smooth curve on $Q(M, \hat{M})$ tangent to $\mathcal{D}_{\mathrm{R}}$. Then $A(t) = (P^{\hat{\nabla}})_0^t(\hat{\gamma}) \circ A(0) \circ (P^{\nabla})_t^0(\gamma)$ and clearly $A(t)^T = (P^{\nabla})_0^t(\gamma) \circ A(0)^T \circ (P^{\hat{\nabla}})_t^0(\hat{\gamma})$, since the g-transpose of $(P^{\nabla})_t^0(\gamma)$ is $(P^{\nabla})_0^t(\gamma)$ and the $\hat{g}$-transpose of $(P^{\hat{\nabla}})_0^t(\hat{\gamma})$ is $(P^{\hat{\nabla}})_t^0(\hat{\gamma})$.

By assumption, $n \leq \hat{n}$, we have for every $q = (x, \hat{x}; A) \in Q(M, \hat{M})$ and for all $X, Y \in T|_x M$ that $g(X,Y) = \hat{g}(AX, AY) = g(A^T AX, Y)$. This means exactly that $A^T A = \mathrm{id}_{T|_x M}$, for all $q = (x, \hat{x}; A) \in Q(M, \hat{M})$, and so, since $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$, $t \in [0,1]$, we have that $A(t)^T \dot{\hat{\gamma}}(t) = A(t)^T A(t) \dot{\gamma}(t) = \dot{\gamma}(t)$.

Thus we have shown that $\tau(q(t))$, $t \in [0,1]$, is the $\widehat{\mathcal{D}_{\mathrm{R}}}$ rolling curve along $\hat{\gamma}$ passing through $\tau(q_0)$, whence $\tau_* \dot{q}(0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \tau(q(t)) \in \widehat{\mathcal{D}_{\mathrm{R}}}|_{\tau(q_0)}$. Since $q(t)$ was arbitrary rolling curve through q_0, we have that $\tau_* \mathcal{D}_{\mathrm{R}}|_{q_0} \subset \widehat{\mathcal{D}_{\mathrm{R}}}|_{\tau(q_0)}$.

□

Remark 11.5 If $n = \hat{n}$, the for dimensional reasons, $\tau_* \mathcal{D}_{\mathrm{R}} = \widehat{\mathcal{D}_{\mathrm{R}}}$. Hence the control systems $(Q(M, \hat{M}), \mathcal{D}_{\mathrm{R}})$ and $(Q(\hat{M}, M), \widehat{\mathcal{D}_{\mathrm{R}}})$ are equivalent. If moreover $M, \hat{M}$ are oriented, it is clear that τ restricts to a diffeomorphism $Q_+(M, \hat{M}) \to Q_+(\hat{M}, M)$.

We can now strengthen somewhat Proposition 10.7.

Proposition 11.6 If $(\hat{M}, \hat{g})$ is complete, then for every $q_0 = (x_0, \hat{x}_0; A_0) \in T^* M \otimes T\hat{M}$ and every a.c. curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$, the rolling curve $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)$ exists and is defined for all $t \in [0,1]$.

Proof. Since $(\hat{M}, \hat{g})$ is complete, Theorem IV.4.1 in [15] implies that for every $\hat{x} \in \hat{M}$ and every a.c. $\hat{\Gamma} : [0,1] \to T|_{\hat{x}}\hat{M}$, there exists a unique a.c. $\hat{\gamma} : [0,1] \to \hat{M}$ such that $\Lambda^{\hat{\nabla}}_{\hat{x}}(\hat{\gamma}) = \hat{\Gamma}(t)$ for all $t \in [0,1]$. The result follows by the argument in the proof of Proposition 10.7, where one is now able to take $T = 1$. □

12 On the Integrability of $\mathcal{D}_{\mathrm{R}}$

In this section we address the following question: given $q_0 \in T^*M \otimes T\hat{M}$ (resp. $q_0 \in Q(M, \hat{M})$), under what conditions on (M, ∇), $(\hat{M}, \hat{\nabla})$ (resp. (M, g), $(\hat{M}, \hat{g})$) there exists an integral manifold of $\mathcal{D}_{\mathrm{R}}$ passing through q_0? It turns out that this question is intimately related to the existence of (local) affine maps between (M, ∇) and $(\hat{M}, \hat{\nabla})$, a result known as Cartan-Ambrose-Hicks theorem (see [5, 19, 23]).

Following result characterizes the integral manifolds of $\mathcal{D}_{\mathrm{R}}$ in $T^*M \otimes T\hat{M}$.

Theorem 12.1 Let (M, ∇), $(\hat{M}, \hat{\nabla})$ be affine manifolds and $q_0 \in T^*M \otimes T\hat{M}$. The following conditions are equivalent:

(i) An orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold of $\mathcal{D}_{\mathrm{R}}$.

(ii) $R^{\mathsf{Rol}}|_q = 0$ and $T^{\mathsf{Rol}}|_q = 0$ for all $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$.

(iii) There exists, for every $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, a connected open neighbourhood $U \subset M$ of x and a unique affine map $\phi : (U, \nabla) \to (\hat{M}, \hat{\nabla})$ such that $\phi_*|_q = A$.

If moreover, M is simply connected and $(\hat{M}, \hat{\nabla})$ is geodesically complete, one can take in (iii), $U = M$.

Proof. For the ease of notation, we write in this proof $\pi_M := \pi_{T^*M \otimes T\hat{M}, M}$, $\pi_{\hat{M}} := \pi_{T^*M \otimes T\hat{M}, \hat{M}}$.

(iii)⟹(ii) Let $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and ϕ be an affine map $\phi : (U, \nabla) \to (\hat{M}, \hat{\nabla})$ such that $\phi_*|_{q_1} = A_1$, where U is a neighbourhood of x in M. Hence, for all smooth curves γ such that $\gamma(0) = x \in U$, it is true that

$$\phi_*|_{\gamma(t)} \circ (P^{\nabla})^t_0(\gamma) = (P^{\hat{\nabla}})^t_0(\phi \circ \gamma) \circ \phi_*|_x,$$

which implies that $\mathscr{L}_{\mathrm{R}}(\dot{\gamma}(0))|_{\phi_*|_x} = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0(\phi_*|_{\gamma(t)})$. Thus $T|_qN = \mathcal{D}_{\mathrm{R}}|_q$, for all $q \in N$, where $N := \{\phi_*|_x \mid x \in U\}$ is the image of the local π_M-section, $x \mapsto \phi_*|_x$. Because N is thus an integral manifold of $\mathcal{D}_{\mathrm{R}}$ passing through $q_1 = \phi_*|_{x_1}$, it follows that for all $X, Y \in \mathrm{VF}(M)$, we have $[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]|_{q_1} \in \mathcal{D}_{\mathrm{R}}|_{q_1}$ and hence the condition (ii) follows from Proposition 4.6 in Paper D.

(ii)⟹(i) The assumption means that the vector fields $\mathscr{L}_{\mathrm{R}}(X)$, $X \in \mathrm{VF}(M)$, restricted to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ form an involutive set of vector fields in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Restricting $\mathcal{D}_{\mathrm{R}}$ onto $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, it follows that $\mathcal{D}_{\mathrm{R}}$ is involutive, hence integrable by Frobenius theorem, distribution on that orbit manifold. It follows that if N is an integral manifold of this restricted $\mathcal{D}_{\mathrm{R}}$ through q_0, then $N = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and thus (i) follows.

(i)⟹(iii) Since $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, it is enough to prove the existence of ϕ in the case where $q = q_0$. Now that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold of $\mathcal{D}_{\mathrm{R}}$, and since $(\pi_M)_*\mathscr{L}_{\mathrm{R}}(X)|_q = X$ for all $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, $X \in T|_xM$, we have that $\pi_M|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to M$, i.e. $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$, is a local diffeomorphism (onto an open subset $\pi_M(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$ of M).

Choose connected open neighbourhoods $O \subset \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ of q_0 and $U \subset M$ of x_0 such that $\pi_M|_O : O \to U$ is a diffeomorphism and define $\phi := \pi_{\hat{M}} \circ (\pi_M|_O)^{-1} : U \to \hat{M}$. We claim that $\phi_*|_{x_0} = A_0$ and that ϕ is an affine map $(U, \nabla) \to (\hat{M}, \hat{\nabla})$.

For the first of these claims, observe that if $x \in U$, $X \in T|_x M$, then $(\pi_M|_O)_*^{-1}(X) \in \mathcal{D}_{\mathrm{R}}|_{(\pi_M|_O)^{-1}(x)}$ and since $(\pi_M)_*(\pi_M|_O)_*^{-1}(X) = X$ and $(\pi_M)_*\mathscr{L}_{\mathrm{R}}(X)|_{(\pi_M|_O)^{-1}(x)} = X$, we have that

$$(\pi_M|_O)_*^{-1}(X) = \mathscr{L}_{\mathrm{R}}(X)|_{(\pi_M|_O)^{-1}(x)}, \quad \forall x \in U.$$

In particular $\phi_*|_{x_0}(X) = (\pi_{\hat{M}})_*\mathscr{L}_{\mathrm{R}}(X)|_{q_0} = A_0 X$, since $(\pi_M|_O)^{-1}(x_0) = q_0$. Moreover, composing the above equation with $(\pi_{\hat{M}})_*$, we have

$$\phi_*|_x = (\pi_M|_O)^{-1}(x), \quad \forall x \in U.$$

For the second claim, consider an arbitrary curve $\gamma : [0,1] \to U$ such that $\gamma(0) \in U$. By what was shown above, $((\pi_M|_O)^{-1})_*(\dot{\gamma}(t)) = \mathscr{L}_{\mathrm{R}}(\dot{\gamma}(t))|_{(\pi_M|_O)^{-1}(\gamma(t))}$ for all $t \in [0,1]$, i.e. $q(t) = (\gamma(t), \hat{\gamma}(t); A(t)) := (\pi_M|_O)^{-1}(\gamma(t))$ is an integral curve of $\mathcal{D}_{\mathrm{R}}$. It follows that $\hat{\gamma}(t) = \phi(\gamma(t))$ for $t \in [0,1]$ and therefore, by Lemma 10.4,

$$(\pi_M|_O)^{-1}(\gamma(t)) = A(t) = (P^{\hat{\nabla}})_0^t(\phi \circ \gamma) \circ (\pi_M|_O)^{-1}(\gamma(0)) \circ (P^{\nabla})_t^0(\gamma), \quad t \in [0,1]$$

and hence, since $\phi_*|_x = (\pi_M|_O)^{-1}(x)$, for all $x \in U$,

$$\phi_*|_{\gamma(t)} \circ (P^{\nabla})_0^t(\gamma) = (P^{\hat{\nabla}})_0^t(\phi \circ \gamma) \circ \phi_*|_{\gamma(0)}.$$

The curve γ being arbitrary in U, the above formula proves that ϕ is indeed an affine map $(U, \nabla) \to (\hat{M}, \hat{\nabla})$ and thus (iii) has been shown.

We now prove the global version of the theorem, i.e. that one can take $U = M$ under the assumptions that M is simply connected and $(\hat{M}, \hat{\nabla})$ is geodesically complete. By Proposition 10.12, we have that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to M$ is a (surjective) bundle map. Since $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is $\mathcal{D}_{\mathrm{R}}$-integral manifold, we know that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is a local diffeomorphism.

We aim to show that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is a covering map. Equip $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ with connection D induced from ∇ by the local diffeomorphism $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$. It follows that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is an affine map $(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0), D) \to (M, \nabla)$. Moreover, $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and M are connected. To prove the claim, we will to use Lemma 3 in [19], which tells that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is a covering map, if we can prove the following: if $\Gamma : J \to \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is a D-geodesic with maximal interval of definition $J \subset \mathbb{R}$, $0 \in J$, then the maximal interval of definition of the ∇-geodesic $\gamma := \pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M} \circ \Gamma$ is J as well.

Suppose that the maximal interval of definition of the geodesic γ is $I \subset \mathbb{R}$, $0 \in \mathbb{R}$. Trivially $J \subset I$. By Proposition 10.7, the rolling curve $q(t) = q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)$ exists for all $t \in I$. But for all $t \in J$ we have $\dot{\Gamma}(t) \in \mathcal{D}_{\mathrm{R}}$, because $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is a $\mathcal{D}_{\mathrm{R}}$-integral manifold, while we also have $(\pi_M \circ \Gamma)(t) = \gamma(t) = (\pi_M \circ q)(t)$, $t \in J$, so we can conclude that $q(t) = \Gamma(t)$ for all $t \in J$. But it is clear that $q(t)$, $t \in I$, is a D-geodesic, whence we obtain that $I = J$.

Since by Lemma 3 in [19] the map $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is a covering map and since M is simply connected, we have that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is bijective. But $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is also a local affine diffeomorphism, hence it is a global affine diffeomorphism. Thus defining as before $\phi := \pi_{\hat{M}} \circ (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1} : M \to \hat{M}$, we have the desired global affine map $(M, \nabla) \to (\hat{M}, \hat{\nabla})$ such that $\phi_*|_{x_0} = A_0$. This completes the proof. □

Remark 12.2 In [19], Theorem 4, one considers, for a given $q_0 = (x_0, \hat{x}_0; A_0) \in T^*M \otimes T\hat{M}$, the following condition:

(iv) For every broken geodesic γ in (M, ∇) such that $\gamma(0) = x_0$, it holds that $R^{\mathsf{Rol}}|_{q_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0)(t)} = 0$ and $T^{\mathsf{Rol}}|_{q_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0)(t)} = 0$ for all t in the domain of definition of $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(\cdot)$.

Recall that a (continuous!) path $\gamma : [a, b] \to M$ in an affine manifold (M, ∇) is called a broken geodesic, if there is a partition $t_0 = a < t_1 < \cdots < t_m = b$, for some $m \in \mathbb{N}$ such that $\gamma|_{[t_i,t_{i+1}]}$ is a ∇-geodesic, for all $i = 0, \ldots, m-1$.

To see that this is equivalent to (ii) of the above theorem, it suffices to observe that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ which is by Proposition 10.7 and by the orbit theorem 3.1,

$$\begin{aligned}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = \{q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) \mid\ & \gamma \text{ piecewise smooth curve } \gamma(0) = x_0, \\ & t \text{ in domain of definition of } q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(\cdot)\},\end{aligned}$$

is the same set as if one replaced on the right hand side the expression "piecewise smooth curve" by "broken geodesic". This follows again from the orbit theorem 3.1 once one observes that for all $q \in T^*M \otimes T\hat{M}$,

$$\mathcal{D}_{\mathrm{R}}|_q = \{\mathscr{L}_{\mathrm{R}}(X)|_q \mid X \text{ is a local geodesic vector field in } (M, \nabla)\},$$

where X is called a local geodesic vector field of (M, ∇) if it is defined in some open subset of M all of its integral curves are ∇-geodesics.

Remark 12.3 For Riemannian manifolds (M, g), $(\hat{M}, \hat{g})$ of the same dimension n, a more extensive list of equivalent characterizations of existence of integral manifolds of $\mathcal{D}_{\mathrm{R}}$ is given in Theorem 5.2 in Paper C.

13 Model of Rolling without Spinning

Remark 13.1 It is well known that $\pi_{T^*M\otimes T\hat{M}} : T^*M \otimes T\hat{M} \to M \times \hat{M}$ is a bundle, and so we don't write it as a separate lemma. Clearly, its fiber $(\pi_{T^*M\otimes T\hat{M}})^{-1}(x, \hat{x})$ of $(x, \hat{x})$ is $T^*|_x M \otimes T|_{\hat{x}}\hat{M}$.

Definition 13.2 Let (M, ∇), $(\hat{M}, \hat{\nabla})$ be affine manifolds. We say that an absolutely continuous curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [a, b]$, in $T^*M \otimes T\hat{M}$ describes the *rolling of* (M, ∇), $(\hat{M}, \hat{\nabla})$ *without spinning*, if for almost all $t \in [a, b]$, the *no-spinning condition* is satisfied: $A(t)\nabla_{\dot{\gamma}(t)}X(t) = \hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(t)X(t))$ for all vector fields $X(t)$ along γ. In this case, we call $q(t)$, $t \in [a, b]$, a *no-spinning curve*.

With respect to Definition 10.3, in rolling with slipping the condition (i) of that definition is not imposed. The analog of Lemma 10.4 is given next (see Lemma 3.6 case (i) in Paper D).

Lemma 13.3 An a.c. curve $q(t)$, $t \in [a,b]$, in $T^*M \otimes T\hat{M}$ is a no-spinning curve if and only if $A(t) = (P^{\nabla})^t_a(\hat{\gamma}) \circ A(a) \circ (P^{\nabla})^a_t(\gamma)$, for all $t \in [a,b]$.

Remark 13.4 In Paper D we used the notation $\mathscr{L}_{\overline{\nabla}}$ (resp. $\mathcal{D}_{\overline{\nabla}}$) instead of $\mathscr{L}_{\mathrm{NS}}$ (resp. $\mathcal{D}_{\mathrm{NS}}$), where in the former $\overline{\nabla}$ refers to the connection induced on $T^*M \otimes T\hat{M}$ by the product connection $\nabla \times \hat{\nabla}$ on $M \times \hat{M}$ while in the latter notation the letters 'NS' refer to 'No-Spinning' condition. We stick here with the latter notation.

Here is the analog of Proposition 10.5 (see also Lemma 3.6 in Paper D), which, in particular, reveals the signification of the $\mathcal{D}_{\mathrm{NS}}$ distribution (Definition 3.4) with respect to the notion of a no-spinning curves.

Proposition 13.5
(i) The map $(\pi_{T^*M\otimes T\hat{M}})_*$ restricted to $\mathcal{D}_{\mathrm{NS}}|_q$ gives an isomorphism $\mathcal{D}_{\mathrm{NS}}|_q \to T|_x M \times T|_{\hat{x}}\hat{M}$, its inverse map being $(X,\hat{X}) \mapsto \mathscr{L}_{\mathrm{NS}}(X,\hat{X})|_q$. Consequently, $\mathcal{D}_{\mathrm{NS}}$ has constant rank $n + \hat{n} = \dim M + \dim \hat{M}$.
(ii) If $X \in \mathrm{VF}(M), \hat{X} \in \mathrm{VF}(\hat{M})$, then the vector field $\mathscr{L}_{\mathrm{NS}}(X,\hat{X})$ on $T^*M \otimes T\hat{M}$ defined by $q = (x,\hat{x};A) \mapsto \mathscr{L}_{\mathrm{NS}}(X|_x,\hat{X}|_{\hat{x}})|_q$ is smooth. Consequently, the rolling distribution $\mathcal{D}_{\mathrm{NS}}$ is smooth.
(iii) An a.c. curve $q(t) = (\gamma(t),\hat{\gamma}(t);A(t))$, $t \in [a,b]$, in $T^*M \otimes T\hat{M}$ is a no-spinning curve if and only if it is tangent to $\mathcal{D}_{\mathrm{NS}}$.

As an immediate corollary to Lemma 13.3 we have a result analogous to Proposition 10.13, that implies that $(T^*M \otimes T\hat{M}, \mathcal{D}_{\mathrm{NS}})$ is never completely controllable.

Proposition 13.6 Let $r_k(M,\hat{M})$ be as in Proposition 10.13. Then for all $k = 0,1,\ldots,\min\{n,\hat{n}\}$,

$$\mathcal{O}_{\mathcal{D}_{\mathrm{NS}}}(q) \subset r_k(M,\hat{M}), \quad \forall q \in r_k(M,\hat{M}).$$

Of course, since $\mathcal{D}_{\mathrm{R}} \subset \mathcal{D}_{\mathrm{NS}}$, the above proposition actually implies Proposition 10.13.

For the rest of the section, we restrict to the case where (M,g), $(\hat{M},\hat{g})$ are connected Riemannian manifolds and ∇, $\hat{\nabla}$ are the respective Levi-Civita connections.

Definition 13.7 We write $\pi_{Q(M,\hat{M})} := \pi_{T^*M\otimes T\hat{M}}|_{Q(M,\hat{M})} : Q(M,\hat{M}) \to M \times \hat{M}$. If $n = \hat{n}$ and $M, \hat{M}$ are oriented, we also set $\pi_{Q_+(M,\hat{M})} := \pi_{T^*M\otimes T\hat{M}}|_{Q_+(M,\hat{M})}$.

We begin with the result about the bundle structure of $\pi_{Q(M,\hat{M})}$.

Proposition 13.8 The map $\pi_{Q(M,\hat{M})}$ is a bundle whose typical fiber is diffeomorphic to $O_n(\mathbb{R}^{\hat{n}})$. In particular, if $n = \hat{n}$ and $M, \hat{M}$ are oriented, then the typical fiber of $\pi_{Q_+(M,\hat{M})}$ is diffeomorphic to $\mathrm{SO}(n)$.

Proof. The latter claim has already been shown to be true at the end of the proof of Proposition 11.3, and to prove that $\pi_{Q(M,\hat{M})}$ is a bundle in general, we essentially repeat the argument.

Given $(x_0, \hat{x}_0) \in M \times \hat{M}$, take any g- and $\hat{g}$-orthonormal frames $F = (X_1, \dots, X_n)$ and $\hat{F} = (\hat{X}_1, \dots, \hat{X}_{\hat{n}})$ defined, respectively, on some open neighbourhoods U and $\hat{U}$ of x_0 and $\hat{x}_0$. For $q = (x, \hat{x}; A) \in (\pi_{Q(M,\hat{M})})^{-1}(U \times \hat{U})$, define $G_{F,\hat{F}}(A)$ to be the $\hat{n} \times n$-matrix whose i-th row and j-th column is $\hat{g}(\hat{X}_i|_{\hat{x}}, AX_j|_x)$ and set

$$\begin{aligned}
&\tau_{F,\hat{F}} : (\pi_{Q(M,\hat{M})})^{-1}(U \times \hat{U}) \to (U \times \hat{U}) \times \mathcal{M}(\hat{n} \times n);\\
&\tau_{F,\hat{F}}(x, \hat{x}; A) = \big((x, \hat{x}), G_{F,\hat{F}}(A)\big),
\end{aligned}$$

where $\mathcal{M}(\hat{n} \times n)$ is the space of $\hat{n} \times n$-real matrices. It is easy to see that $\tau_{F,\hat{F}}$ is smooth, injective and its image is $(U \times \hat{U}) \times O_n(\mathbb{R}^{\hat{n}})$. Moreover, its inverse map $\tau_{F,\hat{F}}^{-1}$: $(U \times \hat{U}) \times O_n(\mathbb{R}^{\hat{n}}) \to (\pi_{Q(M,\hat{M})})^{-1}(U \times \hat{U})$ is given by

$$\tau_{F,\hat{F}}^{-1}((x, \hat{x}), B) = \Big(x, \hat{x}; \sum_{j=1}^{n} \sum_{i=1}^{\hat{n}} B_{ij} g(\cdot, X_j|_x) \hat{X}_i\Big),$$

where B_{ij} is B's i-th row and j-th column. Obviously, $\tau_{F,\hat{F}}^{-1}$ is smooth as well. □

Remark 13.9 One can use the above proposition and the fact that $O_n(\mathbb{R}^{\hat{n}})$ is connected, if $n \neq \hat{n}$, to give a slightly different proof of case (iii) of Proposition 11.3 for $n \neq \hat{n}$.

It follows immediately from the definition of $\mathcal{D}_{\mathrm{NS}}$ and the fact that parallel transport w.r.t. a Levi-Civita connection is an isometry of tangent spaces, that the following holds.

Lemma 13.10 For every $q \in Q(M, \hat{M})$, we have $\mathcal{D}_{\mathrm{NS}}|_q \subset T|_q Q(M, \hat{M})$. Therefore, $\mathcal{D}_{\mathrm{NS}}$ restricts to a smooth distribution on $Q(M, \hat{M})$ of rank $n + \hat{n}$ which we still denote by $\mathcal{D}_{\mathrm{NS}}$ and call the no-spinning distribution on $Q(M, \hat{M})$.

We will end this section with a converse result to Theorem 7.5.

Proposition 13.11 Suppose that (M, g) is oriented, connected, $n = \dim M$, and that $(\hat{M}, \hat{g}) = \mathbb{R}^n$ is the Euclidean n-plane (with natural orientation). Then there is a natural $\mathrm{SO}(n)$-principal bundle structure on $\pi_{Q+(M,\hat{M})}$ that renders $\mathcal{D}_{\mathrm{NS}}$ a principal bundle connection.

Proof. For $q = (x, \hat{x}; A) \in Q(M, \hat{M})$ and $B \in \mathrm{SO}(n)$, we define $\mu(B, q) := (x, \hat{x}; B \circ A)$, where B is interpreted as a map $T|_{\hat{x}}\mathbb{R}^n \to T|_{\hat{x}}\mathbb{R}^n$.

If $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [0, 1]$, is any curve tangent to $\mathcal{D}_{\mathrm{NS}}$ through $q_0 = (x_0, \hat{x}_0; A_0)$, then $A(t) = A(0) \circ (P^{\nabla})^0_t(\gamma)$, since $(P^{\hat{\nabla}})^0_t(\hat{\gamma})$ is just (or is identified with) the identity map in $\mathbb{R}^n$. Therefore $\mu(B, q(t)) = (\gamma(t), \hat{\gamma}(t); B \circ A(0) \circ (P^{\nabla})^0_t(\gamma))$, and so $\mu(B, q(t))$, $t \in [0, 1]$, is a curve tangent to $\mathcal{D}_{\mathrm{NS}}$ and passing through $\mu(B, q_0)$. This implies that $(\mu_B)_* \dot{q}(0) = \mathscr{L}_{\mathrm{NS}}(\dot{\gamma}(0), \dot{\hat{\gamma}}(0))|_{\mu(B,q_0)}$, and hence $(\mu_B)_* \mathcal{D}_{\mathrm{NS}}|_{q_0} = \mathcal{D}_{\mathrm{NS}}|_{\mu(B,q_0)}$, where $\mu_B : Q(M, \hat{M}) \to Q(M, \hat{M})$; $\mu_B(q) = \mu(B, q)$. □

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Part IV

Papers

Summary

A Rolling Manifolds and Controllability: the 3D case

Joint work with Y. Chitour
(submitted)

Rolling Manifolds and Controllability: the 3D case*

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August 22, 2011

Abstract

In this paper, we consider the rolling (R) of one smooth connected complete Riemannian manifold (M, g) onto another one $(\hat{M}, \hat{g})$ of equal dimension $n \geq 2$. The rolling problem (R) corresponds to the situation where there is no relative spin or slip of one manifold with respect to the other one. Since the manifolds are not assumed to be embedded into an Euclidean space, we provide an intrinsic description of the two constraints ”without spinning” and ”without slipping” in terms of the Levi-Civita connections ∇^g and $\nabla^{\hat{g}}$. For that purpose, we recast the the rolling problems within the framework of geometric control and associate to it a distribution and a control system. We then address for the rolling problem the issue of complete controllability. We first provide basic global properties for the reachable set and investigate the associated Lie bracket structure. In particular, we point out the role played by a curvature tensor defined on the state space, that we call the *rolling curvature*. When the two manifolds are three-dimensional, we provide a complete local characterization of the reachable sets and, in particular, we identify necessary and sufficient conditions for the existence of a non open orbit. Besides the trivial case where the manifolds (M, g) and $(\hat{M}, \hat{g})$ are (locally) isometric, we show that (local) non controllability occurs if and only if (M, g) and $(\hat{M}, \hat{g})$ are either warped products or contact manifolds with additional restrictions that we precisely describe.

Contents

*The work of the first author is supported by the ANR project GCM, program “Blanche”, (project number NT09_504490) and the DIGITEO-Région Ile-de-France project CONGEO. The work of the second author is supported by Finnish Academy of Science and Letters.

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1 Introduction

In this paper, we study the rolling of a manifold over another one. Unless otherwise precised, manifolds are smooth, connected, oriented, of finite dimension $n \geq 2$, endowed with a complete Riemannian metric. The rolling is assumed to be without spinning nor slipping and we refer to it as the rolling (R). When both manifolds are isometrically embedded into an Euclidean space, the rolling problem is classical in differential geometry (see [32]), through the notions of "development of a manifold" and "rolling maps". To get an intuitive grasp of the problem, consider the rolling problem (R) of a 2D convex surface S_1 onto another one S_2 in the euclidean space $\mathbb{R}^3$, for instance the plate-ball problem, i.e., a sphere rolling onto a plane in $\mathbb{R}^3$, (cf. [15] and [24]). The two surfaces are in contact i.e., they have a common tangent plane at the contact point and, equivalently, their exterior normal vectors are opposite at the contact point. If $\gamma : [0,T] \to S_1$ is a C^1 regular curve on S_1, one says that S_1 rolls onto S_2 along γ without spinning nor slipping if the following holds. The curve traced on S_1 by the contact point is equal to γ and let $\hat{\gamma} : [0,T] \to S_2$ be the curve traced on S_2 by the contact point. At time $t \in [0,T]$, the relative orientation of S_2 with respect to S_1 is measured by the angle $\theta(t)$ between $\dot{\gamma}(t)$ and $\dot{\hat{\gamma}}(t)$ in the common tangent plane at the contact point. The state space Q of the rolling problem is therefore five dimensional since a point in Q is defined by fixing a point on S_1, a point on S_2 and an angle in S^1, the unit circle. The no-slipping condition says that $\dot{\hat{\gamma}}(t)$ is equal to $\dot{\gamma}(t)$ rotated by the angle $\theta(t)$ and the no-spinning condition characterizes $\dot{\theta}(t)$ in term of the surface elements at $\gamma(t)$ and $\hat{\gamma}(t)$ respectively. Then, once a point on S_2 and an angle are chosen at time $t = 0$, the curves $\hat{\gamma}$ and θ are uniquely determined. The most basic issue in geometric control theory linked to the rolling problem (R) is that of *controllability* i.e., to determine, for two given points q_{init} and q_{final} in the state space Q, if there exists a curve γ so that the rolling of S_1 onto S_2 along γ steers the system from q_{init} to q_{final}. If this is the case for every points q_{init} and q_{final} in Q, then the rolling of S_1 onto S_2 is said to be *completely controllable*.

If the manifolds rolling on each other are two-dimensional, then the controllability issue is well-understood thanks to the work of [3], [6] and [19] especially. For instance, in the simply connected case, the rolling (R) is completely controllable if and only if the manifolds are not isometric. In the case where the manifolds are isometric, [3] also provides a description of the reachable sets in terms of isometries between the manifolds. In particular, these reachable sets are immersed submanifolds of Q of dimension either 2 or 5. In case the manifolds rolling on each other are isometric convex surfaces, [19] provides a beautiful description of a two dimensional reachable set: consider the initial configuration given by two (isometric) surfaces in contact so that one is the image of the other one by the symmetry with respect to the (common) tangent plane at the contact point. Then, this symmetry property (chirality) is preserved along the rolling (R). Note that if the (isometric) convex surfaces are not spheres nor planes, the reachable set starting at a contact point where the Gaussian curvatures are distinct, is open (and thus of dimension 5).

From a robotics point of view, once the controllability is well-understood, the next issue to address is that of *motion planning*, i.e., defining an effective procedure that produces, for every pair of points $(q_{\text{init}},\ q_{\text{final}})$ in the state space Q, a curve $\gamma_{q_{\text{init}},q_{\text{final}}}$ so that the rolling of S_1 onto S_2 along $\gamma_{q_{\text{init}},q_{\text{final}}}$ steers the system from

q_{init} to q_{final}. In [8], an algorithm based on the continuation method was proposed to tackle the rolling problem (R) of a strictly convex compact surface onto an Euclidean plane. That algorithm was also proved in [8] to be convergent and it was numerically implemented in [1] (see also [20] for another algorithm).

The rolling problem (R) is traditionally presented by isometrically embedding the rolling manifolds M and $\hat{M}$ in an Euclidean space (cf. [32], [13]) since it is the most intuitive way to provide a rigorous meaning to the notions of relative spin (or twist) and relative slip of one manifold with respect to the other one. However, the rolling model will depend in general on the embedding. For instance, rolling two 2D spheres of different radii on each other can be isometrically embedded in (at least) two ways in $\mathbb{R}^3$: the smaller sphere can roll onto the bigger one either inside of it or outside. Then one should be able to define rolling without having to resort to any isometric embedding into an Euclidean space. To be satisfactory, that *intrinsic* formulation of the rolling should also allow one to address at least the controllability issue.

The first step towards an intrinsic formulation of the rolling starts with an intrinsic definition of the state space Q. For $n \geq 3$, the relative orientation betwen two manifolds is defined (in coordinates) by an element of $SO(n)$. Therefore the state space Q is locally diffeomorphic to neighborhoods of $M \times \hat{M} \times \mathrm{SO}(n)$ and thus of dimension $2n+n(n-1)/2$. There are two main approaches for an intrinsic formulation of the rolling problem (R), first considered by [3] and [6] respectively. Note that the two references only deal with the two dimensional case but it is immediate to generalize them to higher dimensions. In [3], the state space Q is given by

$$Q = \{A : T|_x M \to T|_{\hat{x}} \hat{M} \mid A \text{ o-isometry},\ x \in M,\ \hat{x} \in \hat{M}\},$$

where "o-isometry" means positively oriented isometry, (see Definition 3.1 below) while in [6], one has equivalently

$$Q = (F_{\mathrm{OON}}(M) \times F_{\mathrm{OON}}(\hat{M}))/\Delta,$$

where $F_{\mathrm{OON}}(M)$, $F_{\mathrm{OON}}(\hat{M})$ be the oriented orthonormal frame bundles of (M, g), $(\hat{M}, \hat{g})$ respectively, and Δ is the diagonal right $\mathrm{SO}(n)$-action.

The next step towards an intrinsic formulation consists of using either the parallel transports with respect to ∇^g and $\nabla^{\hat{g}}$ (Agrachev-Sachkov's approach) or alternatively, orthonormal moving frames and the structure equations (Bryant-Hsu's approach) to translate the constraints of no-spinning and no-slipping and derive the admissible curves, i.e., the curves of Q describing the rolling (R), cf. Eq. (12). Finally, one defines either a distribution or a codistribution depending which approach is chosen. In the present paper, we adopt the Agrachev-Sachkov's approach and we construct an n-dimensional distribution $\mathcal{D}_{\mathrm{R}}$ on Q so that the locally absolutely continuous curves tangent to $\mathcal{D}_{\mathrm{R}}$ are exactly the admissible curves for the rolling problem, cf. Definition 3.17. The construction of $\mathcal{D}_{\mathrm{R}}$ comes along with the construction of (local) basis of vector fields, which allow one to compute the Lie algebraic structure associated to $\mathcal{D}_{\mathrm{R}}$. (See also [21, 10] for alternative constructions of of the rolling problem (R).)

We now describe precisely the results of the present paper. In Section 2, are gathered the notations used throughout the paper. The control system associated to the rolling problem (R) is presented in Section 3 by giving a precise definition of the state space Q and of the set of admissible controls, which is equal the set of locally absolutely continuous (l.a.c.) curves on M only. We thus obtain a driftless control systems affine in the control $(\Sigma)_R$ and also provide, in Appendix A, expressions in local coordinates for these control systems.

In Section 3, we construct an n-dimensional distribution, called the rolling distribution $\mathcal{D}_{\mathrm{R}}$ so that its tangent curves coincide with the admissible curves of $(\Sigma)_R$ and we provide (local) basis of vector fields for $\mathcal{D}_{\mathrm{R}}$. We show that the rolling (R) of M over $\hat{M}$ is symmetric to that of $\hat{M}$ over M i.e., the reachable sets are diffeomorphic. The controllability issue turns out to be a delicate one since, in general, there is no "natural" principal bundle structure on $\pi_{Q,M} : Q \to M$ which leaves invariant the rolling distribution $\mathcal{D}_{\mathrm{R}}$. Indeed, if it were the case, then all the reachable sets would be diffeomorphic and this is not true in general (cf. the description of reachable sets of the rolling problem (R) for two-dimensional isometric manifolds). Despite this fact, we prove that each reachable set is a smooth bundle over M (cf. Proposition 4.2). We also have an equivariance property of the reachable sets of $\mathcal{D}_{\mathrm{R}}$ with respect to the (global) isometries of the manifolds M and $\hat{M}$, as well as an interesting result linking the rolling problem (R) for a pair of manifolds M and $\hat{M}$ and the rolling problem (R) associated to Riemannian coverings of M and $\hat{M}$ respectively. As a consequence, we have that the complete controllability for the rolling problem (R) associated to a pair of manifolds M and $\hat{M}$ is equivalent to that of the rolling problem (R) associated to their universal Riemannian coverings. This implies that, as far as complete controllability is concerned, one can assume without loss of generality that M and $\hat{M}$ are simply connected.

We then compute the first order Lie brackets of the vector fields generating $\mathcal{D}_{\mathrm{R}}$ and find that they are (essentially) equal to the vector fields given by the vertical lifts of

$$\mathsf{Rol}(X,Y)(A) := AR(X,Y) - \hat{R}(AX,AY)A, \tag{1}$$

where X, Y are smooth vector fields of M, $q = (x, \hat{x}; A) \in Q$ and $R(\cdot,\cdot)$, $\hat{R}(\cdot,\cdot)$ are the curvature tensors of g and $\hat{g}$ respectively. We call the vertical vector field defined in Eq. (1) the *Rolling Curvature*, cf Definition 4.9 below. Higher order Lie brackets can now be expressed as linear combinations of covariant derivatives of the Rolling Curvature for the vertical part and evaluations on $\hat{M}$ of the images of the Rolling Curvature and its covariant derivatives.

In dimension two, the Rolling Curvature is (essentially) equal to $K^M(x) - K^{\hat{M}}(\hat{x})$, where $K^M(\cdot)$, $K^{\hat{M}}(\cdot)$ are the Gaussian curvatures of M and $\hat{M}$ respectively. At some point $q \in Q$ where $K^M(x) - K^{\hat{M}}(\hat{x}) \neq 0$, one immediately deduces that the dimension of the evaluation at q of the Lie algebra of the vector fields spanning $\mathcal{D}_{\mathrm{R}}$ is equal to five, (the dimension of Q) and thus the reachable set from q is open in Q. From that fact, one has the following alternative: (a) there exists $q_0 \in Q$ so that $K^M - K^{\hat{M}} \equiv 0$ over the reachable set from q_0, yielding easily that M and $\hat{M}$ have the same Riemannian covering space (cf. [3] and [6]); (b) all the reachable sets are open and then the rolling problem (R) is completely controllable. In dimension $n \geq 3$, the rolling curvature cannot be reduced to a scalar and it is seems difficult compute in general the rank of the evaluations of the Lie algebra of the vector fields spanning $\mathcal{D}_{\mathrm{R}}$. We however can derive an easy sufficent condition for complete controllability, reminiscent of the 2D case: if, for every point q Q, the vertical part of T_qQ belongs to the tangent space at q of the reachable set from$q$, then $(\Sigma)_R$ is completely controllable, cf. Proposition 4.18 (see also [10] for a similar result).

Section 5 collects our results for the rolling (R) of three-dimensional Riemannian manifolds. We are able to provide a complete classification of the possible local structures of a non open orbit, and to each of them, to characterize precisely the manifolds (M,g) and $(\hat{M},\hat{g})$ giving rise to such orbits.

Roughly speaking, what we will prove is that the rolling problem (R) *is not* completely controllable i.e. $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ if and only if the Riemannian manifolds (M,g) and $(\hat{M},\hat{g})$ are *locally* of the following types (i.e., in open dense sets):

(i) isometric,

(ii) both are warped products with similar warping functions or

(iii) both are of class $\mathcal{M}_\beta$ with the same $\beta>0$.

Here, the manifolds of class $\mathcal{M}_\beta$ are defined as three-dimensional Riemannian manifolds carrying a contact structure of particular type, as described in [2] and that we recall in Appendix C.1. The possible values of the orbit dimension d of a non open orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ (i.e. $d=\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$) are correspondingly in (i) $d=3$, (ii) $d=6$ or $d=8$ and finally (iii) $d=7$ or $d=8$, where the alternatives in (ii) and (iii) depend on the initial orientation A_0. Consequently, it follows that the possible orbit dimensions for the rolling of 3D manifolds are

$$\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\in\{3,6,7,8,9\},$$

where dimension $d=9$ corresponds to an open orbit (in Q). Note that we do not answer here to the question of global structure of (M,g), $(\hat{M},\hat{g})$ when the rolling problem (R) is not completely controllable. We finally gather in a series of appendices several results either used in the text or directly related to it. In the final appendix, we provide, for the sake of completeness, the classical formulation of the rolling problem (R) as embedded in an Euclidean space.

Before closing the introduction, let us mention that the results obtained in this paper are actually part of a bigger work, cf. [], where the rolling question is addressed as a general framework of comparing the geometry of two Riemannian manifolds. For instance, we have studied the situtation where one of the manifolds has constant curvature. The particular feature of this case lies in the fact that the state space admits a principal bundle structure compatible with the rolling distribution. We also considered the rolling of two Riemannian manifolds of different dimensions.

Acknowledgements. The authors want to thank U. Boscain, E. Falbel, E. Grong and P. Pansu for helpful comments as well as L. Rifford for having organized the conference "New Trends in Sub-Riemannian Geometry" in Nice and where this work was first presented in April 2010.

2 Notations

For any sets A,B,C and $U\subset A\times B$ and any map $F:U\to C$, we write U_a and U^b for the sets defined by $\{b\in B\mid (a,b)\in U\}$ and $\{a\in A\mid (a,b)\in U\}$ respectively. Similarly, let $F_a:U_a\to C$ and $F^b:U^b\to C$ be defined by $F_a(b):=F(a,b)$ and $F^b(a):=F(a,b)$ respectively. For any sets $V_1,\dots,V_n$ the map $\mathrm{pr}_i:V_1\times\dots\times V_n\to V_i$ denotes the projection onto the i-th factor. For a real matrix A, we use A^i_j to denote the real number on the i-th row and j-th column and the matrix A can then be denoted by $[A^i_j]$. If, for example, one has $A^i_j = a_{ij}$ for all i, j, then one uses the notation $A^i_j = (a_{ij})^i_j$ and thus $A = [(a_{ij})^i_j]$. The matrix multiplication of $A = [A^i_j]$ and $B = [B^i_j]$ is therefore given by $AB = \left[\left(\sum_k A^i_k B^k_j\right)^i_j\right]$. Suppose V, W are finite dimensional $\mathbb{R}$-linear spaces, $L : V \to W$ is an $\mathbb{R}$-linear map and $F = (v_i)_{i=1}^{\dim V}$, $G = (w_i)_{i=1}^{\dim W}$ are bases of V, W respectively. The $\dim W \times \dim V$-real matrix corresponding to L w.r.t. the bases F and G is denoted by $\mathcal{M}_{F,G}(L)$. In other words, $L(v_i) = \sum_j \mathcal{M}_{F,G}(L)^j_i w_j$ (corresponding to the right multiplication by a matrix of a row vector). Notice that, if $K : W \to U$ is yet another $\mathbb{R}$-linear map to a finite dimensional linear space U with basis $H = (u_i)_{i=1}^{\dim U}$, then

$$\mathcal{M}_{F,H}(K \circ L) = \mathcal{M}_{G,H}(K)\mathcal{M}_{F,G}(L).$$

If (V, g), (W, h) are inner product spaces with inner products g and h, one defines $L^{T_{g,h}} : W \to V$ as the transpose (adjoint) of A w.r.t g and h i.e., $g(L^{T_{g,h}}w, v) = h(w, Lv)$. With bases F and G as above, one has $\mathcal{M}_{F,G}(L)^T = \mathcal{M}_{G,F}(L^{T_{g,h}})$, where T on the left is the usual transpose of a real matrix i.e., the transpose w.r.t standard Euclidean inner products in $\mathbb{R}^N$, $N \in \mathbb{N}$.

In this paper, by a smooth manifold, one means a smooth finite-dimensional, second countable, Hausdorff manifold (see e.g. [18]). By a smooth submanifold of M, we always mean a smooth embedded submanifold. For any smooth map $\pi : E \to M$ between smooth manifolds E and M, the set $\pi^{-1}(\{x\}) =: \pi^{-1}(x)$ is called the π-fiber over x and it is sometimes denoted by $E|_x$, when π is clear from the context. The set of smooth sections of π is denoted by $\Gamma(\pi)$. The value $s(x)$ of a section s at x is usually denoted by $s|_x$. A smooth manifold M is oriented if there exists a smooth (or continuous) section, defined on all of M, of the bundle of n-forms $\pi_{\bigwedge^n(M)} : \bigwedge^n(M) \to M$ where $n = \dim M$. If not otherwise mentioned, the smooth manifolds considered in this paper are connected and oriented. For a smooth map $\pi : E \to M$ and $y \in E$, let $V|_y(\pi)$ be the set of all $Y \in T|_y E$ such that $\pi_*(Y) = 0$. If π is a smooth bundle, the collection of spaces $V|_y(\pi)$, $y \in E$, defines a smooth submanifold $V(\pi)$ of $T(E)$ and the restriction $\pi_{T(E)} : T(E) \to E$ to $V(\pi)$ is denoted by $\pi_{V(\pi)}$. In this case $\pi_{V(\pi)}$ is a vector subbundle of $\pi_{T(E)}$ over E. For a smooth manifold M, one uses $\mathrm{VF}(M)$ to denote the set of smooth vector fields on M i.e., the set of smooth sections of the tangent bundle $\pi_{T(M)} : T(M) \to M$. The flow of a vector field $Y \in \mathrm{VF}(M)$ is a smooth onto map $\Phi_Y : D \to M$ defined on an open subset D of $\mathbb{R} \times M$ containing $\{0\} \times M$ such that $\frac{\partial}{\partial t}\Phi_Y(t, y) = Y|_{\Phi_Y(t,y)}$ for $(t, y) \in D$ and $\Phi_Y(0, y) = y$ for all $y \in M$. As a default, we will take D to be the maximal flow domain of Y.

For any distribution $\mathcal{D}$ on a manifold M, we use $\mathrm{VF}_{\mathcal{D}}$ to denote the set of vector fields $X \in \mathrm{VF}(M)$ tangent to $\mathcal{D}$ (i.e., $X|_x \in \mathcal{D}|_x$ for all $x \in M$) and we define inductively for $k \geq 2$, $\mathrm{VF}^k_{\mathcal{D}} = \mathrm{VF}^{k-1}_{\mathcal{D}} + [\mathrm{VF}_{\mathcal{D}}, \mathrm{VF}^{k-1}_{\mathcal{D}}]$, where $\mathrm{VF}^1_{\mathcal{D}} := \mathrm{VF}_{\mathcal{D}}$. The Lie algebra generated by $\mathrm{VF}_{\mathcal{D}}$ is denoted by $\mathrm{Lie}(\mathcal{D})$ and it equals $\bigcup_k \mathrm{VF}^k_{\mathcal{D}}$. For any maps $\gamma : [a, b] \to X$, $\omega : [c, d] \to X$ into a set X such that $\gamma(b) = \omega(c)$ we define

$$\omega \sqcup \gamma : [a, b + d - c] \to X; \quad (\omega \sqcup \gamma)(t) = \begin{cases} \gamma(t), & t \in [a, b], \\ \omega(t - b + c), & t \in [b, b + d - c]. \end{cases}$$

A map $\gamma : [a, b] \to X$ is a loop in X based at $x_0 \in X$ if $\gamma(a) = \gamma(b) = x_0$. In the space of loops $[0, 1] \to X$ based at some given point x_0, one defines a group operation ”.”, concatenation, by

$$\omega.\gamma := (t \mapsto \omega(\tfrac{t}{2})) \sqcup (t \mapsto \gamma(\tfrac{t}{2})).$$

This operation gives a group structure on the set of loops of X based at a given point x_0. If N is a smooth manifold and $y \in N$, we use $\Omega_y(N)$ to denote the set of all piecewise C^1-loops $[0,1] \to N$ of N based at y. In particular, $(\Omega_y(N), .)$ is a group.

Given a smooth distribution $\mathcal{D}$ on a smooth manifold M, we call an absolutely continuous curve $c : I \to M$, $I \subset \mathbb{R}$, $\mathcal{D}$-*admissible* if c it is tangent to $\mathcal{D}$ almost everywhere (a.e.) i.e., if for almost all $t \in I$ it holds that $\dot{c}(t) \in \mathcal{D}|_{c(t)}$. For $x_0 \in M$, the endpoints of all the $\mathcal{D}$-admissible curves of M starting at x_0 form the set called $\mathcal{D}$-*orbit through* x_0 and denoted $\mathcal{O}_{\mathcal{D}}(x_0)$. More precisely,

$$\mathcal{O}_{\mathcal{D}}(x_0) = \{c(1) \mid c : [0,1] \to M,\ \mathcal{D}\text{−admissible},\ c(0) = x_0\}. \tag{2}$$

By the Orbit Theorem (see [4]), it follows that $\mathcal{O}_{\mathcal{D}}(x_0)$ is an immersed smooth submanifold of M containing x_0. It is also known that one may restrict to piecewise smooth curves in the description of the orbit i.e.,

$$\mathcal{O}_{\mathcal{D}}(x_0) = \{c(1) \mid c : [0,1] \to M \text{ piecewise smooth and } \mathcal{D}\text{−admissible},\ c(0) = x_0\}.$$

We call a smooth distribution $\mathcal{D}'$ on M a subdistribution of $\mathcal{D}$ if $\mathcal{D}' \subset \mathcal{D}$. An immediate consequence of the definition of the orbit shows that in this case, for all $x_0 \in M$, $\mathcal{O}_{\mathcal{D}'}(x_0) \subset \mathcal{O}_{\mathcal{D}}(x_0)$.

If $\pi : E \to M$, $\eta : F \to M$ are two smooth maps (e.g. bundles), let $C^\infty(\pi, \eta)$ be the set of all bundle maps $\pi \to \eta$ i.e., smooth maps $g : E \to F$ such that $\eta \circ g = \pi$. For a manifold M, let $\pi_{M_{\mathbb{R}}} : M \times \mathbb{R} \to M$ be the projection onto the first factor i.e., $(x,t) \mapsto x$ (i.e., $\pi_{M_{\mathbb{R}}} = \mathrm{pr}_1$). If $\pi : E \to M$, $\eta : F \to M$ are any smooth vector bundles over a smooth manifold M, $f \in C^\infty(\pi, \eta)$ and $u, w \in \pi^{-1}(x)$, one defines the vertical derivative f at u in the direction w by

$$\nu(w)|_u(f) := (D_\nu f)(u)(w) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(u + tw). \tag{3}$$

Here $w \mapsto (D_\nu f)(u)(w) = \nu(w)|_u(f)$ is an $\mathbb{R}$-linear map between fibers $\pi^{-1}(x) \to \eta^{-1}(x)$. In a similar way, in the case of $f \in C^\infty(E)$ and $u, w \in \pi^{-1}(x)$, one defines the π-vertical derivative $\nu(w)|_u(f) := D_\nu f(u)(w) := \frac{\mathrm{d}}{\mathrm{d}t}|_0 f(u+tw)$ at u in the direction w. This definition agrees with the above one modulo the canonical bijection $C^\infty(E) \cong C^\infty(\mathrm{id}_E, \pi_{E_{\mathbb{R}}})$. This latter definition means that $\nu(w)|_u$ can be viewed as an element of $V|_u(\pi)$ and the mapping $w \mapsto \nu(w)|_u$ gives a (natural) $\mathbb{R}$-linear isomorphism between $\pi^{-1}(x)$ and $V|_u(\pi)$ where $\pi(u) = x$. If $\tilde{w} \in \Gamma(\pi)$ is a smooth π-section, let $\nu(\tilde{w})$ be the π-vertical vector field on E defined by $\nu(\tilde{w})|_u(f) = \nu(\tilde{w}|_x)|_u(f)$, where $\pi(u) = x$ and $f \in C^\infty(E)$. The same remark holds also locally.

In the case of smooth manifolds M and $\hat{M}$, $x \in M$, $\hat{x} \in \hat{M}$, we will use freely and without mention the natural inclusions ($\subset$) and isomorphisms ($\cong$): $T|_x M, T|_{\hat{x}}\hat{M} \subset T|_{(x,\hat{x})}(M \times \hat{M}) \cong T|_x M \oplus T|_{\hat{x}}\hat{M}$, $T^*|_x M, T^*|_{\hat{x}}\hat{M} \subset T^*|_{(x,\hat{x})}(M \times \hat{M}) \cong T^*|_x M \oplus T^*|_{\hat{x}}\hat{M}$. An element of $T|_{(x,\hat{x})}(M \times \hat{M}) \cong T|_x(M) \oplus T|_{\hat{x}}(\hat{M})$ with respect to the direct sum splitting is denoted usually by $(X, \hat{X})$, where $X \in T|_x M$, $\hat{X} \in T|_{\hat{x}}\hat{M}$. Sometimes it is even more convenient to write $X + \hat{X} := (X, \hat{X})$ when we make the identifications $(X,0)=X$, $(0,\hat{X})=\hat{X}$. Let (M,g), $(\hat{M},\hat{g})$ be smooth Riemannian manifolds. A map $f:M\to\hat{M}$ is a *local isometry* if it is smooth, surjective and for all $x\in M$, $f_*|_x:T|_xM\to T|_{f(x)}\hat{M}$ is an isometric linear map. A bijective local isometry $f:M\to\hat{M}$ is called an *isometry* and then (M,g), $(\hat{M},\hat{g})$ are said to be *isometric*. In this text we say that two Riemannian manifolds (M,g), $(\hat{M},\hat{g})$ are *locally isometric*, if there is a Riemannian manifold (N,h) and local isometries $F:N\to M$ and $G:N\to\hat{M}$ which are also covering maps i.e. if they are *Riemannian covering maps*. One calls (N,h) a common Riemannian covering space of (M,g) and $(\hat{M},\hat{g})$. Notice that being locally isometric is an equivalence relation in the class of smooth Riemannian manifolds (the fact that we assume F,G to be Riemannian covering maps, and not only local isometries, implies the transitivity of this relation).

The space $\overline{M}=M\times\hat{M}$ is a Riemannian manifold, called the Riemannian product manifold of (M,g), $(\hat{M},\hat{g})$, when endowed with the product metric $\overline{g}:=g\oplus\hat{g}$. One often writes this as $(M,g)\times(\hat{M},\hat{g})$. Let ∇, $\hat{\nabla},\overline{\nabla}$ (resp. $R,\hat{R},\overline{R}$) denote the Levi-Civita connections (resp. the Riemannian curvature tensors) of (M,g), $(\hat{M},\hat{g}),(\overline{M}=M\times\hat{M},\overline{g}=g\oplus\hat{g})$ respectively. From Koszul's formula (cf. [18]), one has

$$\overline{\nabla}_{(X,\hat{X})}(Y,\hat{Y})=(\nabla_XY,\hat{\nabla}_{\hat{X}}\hat{Y}),\tag{4}$$

when $X,Y\in\mathrm{VF}(M)$, $\hat{X},\hat{Y}\in\mathrm{VF}(\hat{M})$ and hence from the definition of the Riemannian curvature tensor

$$\overline{R}((X,\hat{X}),(Y,\hat{Y}))(Z,\hat{Z})=(R(X,Y)Z,\hat{R}(\hat{X},\hat{Y})\hat{Z}),\tag{5}$$

where $X,Y,Z\in T|_xM$, $\hat{X},\hat{Y},\hat{Z}\in T|_{\hat{x}}\hat{M}$. For any (k,m)-tensor field T on M we define ∇T to be the $(k,m+1)$-tensor field such that (see [30], p. 30)

$$(\nabla T)(X_1,\dots,X_m,X)=(\nabla_XT)(X_1,\dots,X_m),\tag{6}$$

$X_1,\dots,X_m,X\in T|_xM$.

The parallel transport of a tensor $T_0\in T^k_m|_{\gamma(0)}(M)$ from $\gamma(0)$ to $\gamma(t)$ along an absolutely continuous curve $\gamma:I\to M$ (with $0\in I$) and with respect to the Levi-Civita connection of (M,g) is denoted by $(P^{\nabla^g})^t_0(\gamma)T_0$. In the notation of the Levi-Civita connection ∇^g (resp. parallel transport P^{∇^g}), the upper index g (resp. ∇^g) referring to the Riemannian metric g (resp. the connection ∇^g) is omitted if it is clear from the context. Let $(\gamma,\hat{\gamma}):I\to M\times\hat{M}$ be a smooth curve on $M\times\hat{M}$ defined on an open real interval I containing 0. If $(X(t),\hat{X}(t)):I\to T(M\times\hat{M})$ is a smooth vector field on $M\times\hat{M}$ along $(\gamma,\hat{\gamma})$ i.e., $(X(t),\hat{X}(t))\in T|_{(\gamma(t),\hat{\gamma}(t))}(M\times\hat{M})$ then one has

$$\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(X,\hat{X})=(\nabla_{\dot{\gamma}(t)}X,\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{X})\tag{7}$$

if the covariant derivatives on the right-hand side are well defined.

If (N,h) is a Riemannian manifold we define $\mathrm{Iso}(N,h)$ to be the (smooth Lie) group of isometries of (N,h) (cf. [30], Lemma III.6.4, p. 118). It is clear that the isometries respect parallel transport in the sense that for any absolutely continuous $\gamma:[a,b]\to N$ and $F\in\mathrm{Iso}(N,g)$ one has (cf. [30], p. 41, Eq. (3.5))

$$F_*|_{\gamma(t)}\circ(P^{\nabla^h})^t_a(\gamma)=(P^{\nabla^h})^t_a(F\circ\gamma)\circ F_*|_{\gamma(a)}.\tag{8}$$

The following result is standard.

Theorem 2.1 Let (N, h) be a Riemannian manifold and for any absolutely continuous $\gamma : [0, 1] \to M$, $\gamma(0) = y_0$, define

$$\Lambda^{\nabla^h}_{y_0}(\gamma)(t) = \int_0^t (P^{\nabla^h})^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s \in T|_{y_0}N, \quad t \in [0, 1].$$

Then the map $\Lambda^{\nabla^h}_{y_0} : \gamma \mapsto \Lambda^{\nabla^h}_{y_0}(\gamma)(\cdot)$ is an injection from the set of absolutely continuous curves $[0, 1] \to N$ starting at y_0 onto an open subset of the Banach space of absolutely continuous curves $[0, 1] \to T|_{y_0}N$ starting at 0.

Moreover, the map $\Lambda^{\nabla^h}_{y_0}$ is a bijection onto the latter Banach space if (and only if) (N, h) is a complete Riemannian manifold.

Remark 2.2 (i) For example, in the case where γ is the geodesic $t \mapsto \exp_{y_0}(tY)$ for $Y \in T|_{y_0}N$, one has

$$\Lambda^{\nabla^h}_{y_0}(\gamma)(t) = tY.$$

(ii) It is directly seen from the definition of $\Lambda^{\nabla^h}_{y_0}$ that it maps injectively (piecewise) C^k-curves, $k = 1, \dots, \infty$, starting at y_0 to (piecewise) C^k-curves starting at 0. Moreover, these correspondences are bijective if (N, h) is complete.

3 State Space, Distributions and Computational Tools

3.1 State Space

3.1.1 Definition of the state space

After [3], [4] we make the following definition.

Definition 3.1 The *state space* $Q = Q(M, \hat{M})$ for the rolling of two n-dimensional *connected, oriented* smooth Riemannian manifolds $(M, g), (\hat{M}, \hat{g})$ is defined as

$$Q = \{A : T|_x M \to T|_{\hat{x}}\hat{M} \mid A \text{ o-isometry}, \ x \in M, \ \hat{x} \in \hat{M}\},$$

where "o-isometry" stands for "orientation preserving isometry" i.e., if $(X_i)_{i=1}^n$ is a positively oriented g-orthonormal frame of M at x then $(AX_i)_{i=1}^n$ is a positively oriented $\hat{g}$-orthonormal frame of $\hat{M}$ at $\hat{x}$.

The linear space of $\mathbb{R}$-linear map $A : T|_x M \to T|_{\hat{x}}\hat{M}$ is canonically isomorphic to the tensor product $T^*|_x M \otimes T|_{\hat{x}}\hat{M}$. On the other hand, by using the canonical inclusions $T^*|_x M \subset T^*|_{(x,\hat{x})}(M \times \hat{M})$, $T|_{\hat{x}}\hat{M} \subset T|_{(x,\hat{x})}(M \times \hat{M})$, the space $T^*|_x M \otimes T|_{\hat{x}}\hat{M}$ is canonically included in the space $T^1_1(M \times \hat{M})|_{(x,\hat{x})}$ of $(1, 1)$-tensors of $M \times \hat{M}$ at $(x, \hat{x})$. These inclusions make $T^*M \otimes T\hat{M} := \bigcup_{(x,\hat{x})\in M\times\hat{M}} T^*|_x M \otimes T|_{\hat{x}}\hat{M}$ a subset of $T^1_1(M \times \hat{M})$ such that $\pi_{T^*M\otimes T\hat{M}} := \pi_{T^1_1(M\times\hat{M})}|_{T^*M\otimes T\hat{M}} : T^*M \otimes T\hat{M} \to M \times \hat{M}$ is a smooth vector subbundle of the bundle of $(1, 1)$-tensors $\pi_{T^1_1(M\times\hat{M})}$ on $M \times \hat{M}$.

The state space $Q = Q(M, \hat{M})$ can be described as a subset of $T^*M \otimes T\hat{M}$ as

$$Q = \{A \in T^*M \otimes T\hat{M}|_{(x,\hat{x})} \mid (x, \hat{x}) \in M \times \hat{M}, \\ \|AX\|_{\hat{g}} = \|X\|_g, \ \forall X \in T|_x M, \ \det(A) = 1\}.$$

In the next subsection, we will show that $\pi_Q := \pi_{T^*M\otimes T\hat{M}}|_Q$ is moreover a smooth subbundle of $\pi_{T^*M\otimes T\hat{M}}$. It is also sometimes convenient to consider the manifold $T^*M \otimes T\hat{M}$ and we will refer to it as the *extended state space* for the rolling. This concept of extended state space naturally makes sense also in the case where M and $\hat{M}$ are not assumed to be oriented (or connected). A point $A \in T^*M \otimes T\hat{M}$ with $\pi_{T^*M\otimes T\hat{M}}(A) = (x, \hat{x})$ (or $A \in Q$ with $\pi_Q(A) = (x, \hat{x})$) will be usually denoted by $(x, \hat{x}; A)$ to emphasize the fact that $A : T|_x M \to T|_{\hat{x}}\hat{M}$. Thus the notation $q = (x, \hat{x}; A)$ simply means that $q = A$.

3.1.2 The Bundle Structure of Q

In this subsection, it is shown that π_Q is a bundle with typical fiber $\mathrm{SO}(n)$.

Definition 3.2 Suppose the vector fields $X_i \in \mathrm{VF}(M)$ (resp. $\hat{X}_i \in \mathrm{VF}(\hat{M})$), $i = 1, \dots, n$ form a g-orthonormal (resp. $\hat{g}$-orthonormal) frame of vector fields on an open subset U of M (resp. $\hat{U}$ of $\hat{M}$). We denote $F = (X_i)_{i=1}^n$, $\hat{F} = (\hat{X}_i)_{i=1}^n$ and for $x \in U$, $\hat{x} \in \hat{U}$ we let $F|_x = (X_i|_x)_{i=1}^n$, $\hat{F}|_{\hat{x}} = (\hat{X}_i|_{\hat{x}})_{i=1}^n$ Then a local trivialization $\tau = \tau_{F,\hat{F}}$ of Q over $U \times \hat{U}$ *induced by* $F, \hat{F}$ is given by

$$\begin{aligned}\tau : \pi_Q^{-1}(U \times \hat{U}) &\to (U \times \hat{U}) \times \mathrm{SO}(n)\\ (x, \hat{x}; A) &\mapsto \big((x, \hat{x}), \mathcal{M}_{F|_x, \hat{F}|_{\hat{x}}}(A)\big),\end{aligned}$$

where $\mathcal{M}_{F|_x, \hat{F}|_{\hat{x}}}(A)_i^j = \hat{g}(AX_i, \hat{X}_j)$ since $AX_i|_x = \sum_j \hat{g}(AX_i|_x, \hat{X}_j|_{\hat{x}})\hat{X}_j|_{\hat{x}}$.

For the sake of clarity, we shall write $\mathcal{M}_{F|_x, \hat{F}|_{\hat{x}}}(A)$ as $\mathcal{M}_{F,\hat{F}}(A)$. Obviously $\|AX\|_{\hat{g}} = \|X\|_g$ for all $X \in T|_x M$ is equivalent to $A^{T_{g,\hat{g}}} A = \mathrm{id}_{T|_x M}$ and we get

$$\mathcal{M}_{F,\hat{F}}(A)^T \mathcal{M}_{F,\hat{F}}(A) = \mathcal{M}_{\hat{F},F}(A^{T_{g,\hat{g}}})\mathcal{M}_{F,\hat{F}}(A) = \mathcal{M}_{F,F}(\mathrm{id}_{T|_x M}) = \mathrm{id}_{\mathbb{R}^n},$$

where T denotes the usual transpose in $\mathfrak{gl}(n)$, the set of Lie algebra of $n \times n$-real matrices. Since $\det \mathcal{M}_{F,\hat{F}}(A) = \det(A) = +1$, one has $\mathcal{M}_{F,\hat{F}}(A) \in \mathrm{SO}(n)$.

Remark 3.3 Notice that the above local trivializations $\tau_{F,\hat{F}}$ of π_Q are just the restrictions of the vector bundle local trivializations

$$(\pi_{T^*(M)\otimes T(\hat{M})})^{-1}(U \times \hat{U}) \to (U \times \hat{U}) \times \mathfrak{gl}(n)$$

of the bundle $\pi_{T^*(M)\otimes T(\hat{M})}$ induced by $F, \hat{F}$ and defined by the same formula as $\tau_{F,\hat{F}}$. In this setting, one does not even have to assume that the local frames F, $\hat{F}$ are g- or $\hat{g}$-orthonormal. Hence π_Q is a smooth subbundle of $\pi_{T^*M\otimes T\hat{M}}$ with Q a smooth submanifold of $T^*M \otimes T\hat{M}$.

Notice that any π_Q-vertical tangent vector (i.e., an element of $V|_q(\pi_Q)$) is of the form $\nu(B)|_q$ for a unique $B \in T^*M \otimes T\hat{M}|_{(x,\hat{x})}$ where $q = (x, \hat{x}; A) \in Q$. The following simple proposition gives the condition when, for a $B \in T^*M \otimes T\hat{M}|_{(x,\hat{x})}$, the vector $\nu(B)|_q \in V|_q(\pi_{T^*M\otimes T\hat{M}})$ is actually tangent to Q i.e., an element of $V|_q(\pi_Q)$.

Proposition 3.4 Let $q = (x, \hat{x}; A) \in Q$ and $B \in T^*(M) \otimes T(\hat{M})|_{(x,\hat{x})}$. Then $\nu(B)|_q$ is tangent to Q (i.e., is an element of $V|_q(\pi_Q)$) if and only if

$$\hat{g}(AX, BY) + \hat{g}(BX, AY) = 0,$$

for all $X, Y \in T|_x M$. Denoting $\overline{T} = T_{g,\hat{g}}$, this latter condition can be stated equivalently as $A^{\overline{T}} B + B^{\overline{T}} A = 0$ or more compactly as $B \in A(\mathfrak{so}(T|_x M))$.

We use $\overline{T}$ to denote the $(g, \hat{g})$-transpose operation $T_{g,\hat{g}}$ in the sequel. The proposition says that $V|_q(\pi_Q)$ is naturally $\mathbb{R}$-linearly isomorphic to $A(\mathfrak{so}(T|_x M))$.

3.2 Distribution and the Control Problem

3.2.1 From Rolling to Distributions

Each point $(x, \hat{x}; A)$ of the state space $Q = Q(M, \hat{M})$ can be viewed as describing a contact point of the two manifolds which is given by the points x and $\hat{x}$ of M and $\hat{M}$, respectively, and an isometry A of the tangent spaces $T|_x M$, $T|_{\hat{x}} \hat{M}$ at this contact point. The isometry A can be viewed as measuring the relative orientation of these tangent spaces relative to each other in the sense that rotation of, say, $T|_{\hat{x}} \hat{M}$ corresponds to a unique change of the isometry A from $T|_x M$ to $T|_{\hat{x}} \hat{M}$. A curve $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ in Q can then be seen as a motion of M against $\hat{M}$ such that at an instant t, $\gamma(t)$ and $\hat{\gamma}(t)$ represent the common point of contact in M and $\hat{M}$, respectively, and $A(t)$ measures the relative orientation of coinciding tangent spaces $T|_{\gamma(t)} M$, $T|_{\hat{\gamma}(t)} \hat{M}$ at this point of contact.

In order to call this motion *rolling*, there are two kinematic constraints that will be demanded (see e.g. [3], [4] Chapter 24, [8]) namely

(i) the *no-spinning* condition;

(ii) the *no-slipping* condition.

In this section, these conditions will be defined explicitly and it will turn out that they are modeled by certain smooth distributions on the state space Q. The subsequent sections are then devoted to the detailed definitions and analysis of the distribution $\mathcal{D}_{\mathrm{NS}}$ and $\mathcal{D}_{\mathrm{R}}$ on the state space Q, the former capturing the no-spinning condition (i) while the latter capturing both of the conditions (i) and (ii).

The first restriction (i) for the motion is that the relative orientation of the two manifolds should not change along motion. This *no-spinning condition* (also known as the no-twisting condition) can be formulated as follows.

Definition 3.5 An absolutely continuous (*a.c.*) curve

$$\begin{aligned} q : \ & I \to Q, \\ & t \mapsto (\gamma(t), \hat{\gamma}(t); A(t)), \end{aligned}$$

defined on some real interval $I = [a, b]$, is said to describe a *motion without spinning* of M against $\hat{M}$ if, for every a.c. curve $[a, b] \to TM$; $t \mapsto X(t)$ of vectors along $t \mapsto \gamma(t)$, we have

$$\nabla_{\dot{\gamma}(t)} X(t) = 0 \implies \hat{\nabla}_{\dot{\hat{\gamma}}(t)} (A(t)X(t)) = 0 \quad \text{for a.e. } t \in [a, b]. \tag{9}$$

(See also [21] for a similar definition.) Note that Condition (9) is equivalent to the following: for a. e. t and all parallel vector fields $X(\cdot)$ along $x(\cdot)$, one has

$$(\overline{\nabla}_{(\dot\gamma(t),\dot{\hat\gamma}(t))}A(t))X(t) = 0.$$

Since the parallel translation $P_0^t(\gamma) : T|_{\gamma(0)}M \to T|_{\gamma(t)}M$ along $\gamma(\cdot)$ is an (isometric) isomorphism (here $X(t) = P_0^t(\gamma)X(0)$), then (9) is equivalent to

$$\overline{\nabla}_{(\dot\gamma(t),\dot{\hat\gamma}(t))}A(t) = 0 \quad \text{for a.e. } t \in [a,b]. \tag{10}$$

The second restriction (ii) is that the manifolds should not slip along each other as they move i.e., the velocity of the contact point should be the same w.r.t both manifolds. This *no-slipping condition* can be formulated as follows.

Definition 3.6 An a.c. curve $q : I \to Q$; $t \mapsto (\gamma(t),\hat\gamma(t);A(t))$, defined on some real interval $I = [a,b]$, is said to describe a *motion without slipping* of M against $\hat M$ if

$$A(t)\dot\gamma(t) = \dot{\hat\gamma}(t) \quad \text{for a.e. } t \in [a,b]. \tag{11}$$

Definition 3.7 An a.c. curve $q : I \to Q$; $t \mapsto (\gamma(t),\hat\gamma(t);A(t))$, defined on some real interval $I = [a,b]$, is said to describe a *rolling motion* i.e., a *motion without slipping or spinning* of M against $\hat M$ if it satisfied both of the conditions (9),(11) (or equivalently (10),(11)). The corresponding curve $t \mapsto (\gamma(t),\hat\gamma(t);A(t))$ that satisfies these conditions is called a *rolling curve*.

It is easily seen that $t \mapsto q(t) = (\gamma(t),\hat\gamma(t);A(t))$, $t \in [a,b]$, is a rolling curve if and only if it satisfies the following driftless control affine system

$$(\Sigma)_R \quad \begin{cases} \dot\gamma(t) = u(t), \\ \dot{\hat\gamma}(t) = A(t)u(t), \qquad\qquad \text{for a.e. } t \in [a,b], \\ \overline{\nabla}_{(u(t),A(t)u(t))}A(t) = 0, \end{cases} \tag{12}$$

where the control u belongs to $\mathcal{U}(M)$, the set of measurable TM-valued functions u defined on some interval $I = [a,b]$ such that there exists a.c. $y : [a,b] \to M$ verifying $u = \dot y$ a.e. on $[a,b]$. Conversely, given any control $u \in \mathcal{U}(M)$ and $q_0 = (x_0,\hat x_0;A_0) \in Q$, a solution $q(\cdot)$ to this control system exists on a subinterval $[a,b']$, $a < b' \le b$ satisfying the initial condition $q(a) = q_0$. The fact that System (12) is driftless and control affine can be seen from its representation in local coordinates (see Eqs. (53)-(55) in Appendix A).

We begin by recalling some basic observations on parallel transport. As is clear, if one starts with a $(1,1)$-tensor $A_0 \in T_1^1|_{(x_0,\hat x_0)}(M \times \hat M)$ and has an a.c. curve $t \mapsto (\gamma(t),\hat\gamma(t))$ on $M \times \hat M$ with $\gamma(0) = x_0$, $\hat\gamma(0) = \hat x_0$, defined on an open interval $I \ni 0$, then the parallel transport $A(t) = P_0^t(\gamma,\hat\gamma)A_0$ exists on I and determines an a.c. curve in $T_1^1(M \times \hat M)$. But now, if A_0 rather belongs to the subspace $T^*M \otimes T\hat M$ or Q of $T_1^1(M \times \hat M)$, it will actually happen that the parallel translate $A(t)$ belongs to this subspace as well for all $t \in I$. This is the content of the next proposition, whose proof is straightforward.

Proposition 3.8 Let $t \mapsto (\gamma(t), \hat{\gamma}(t))$ be an absolutely continuous curve in $M \times \hat{M}$ defined on some real interval $I \ni 0$. Then we have

$$A_0 \in T^*M \otimes TM \Longrightarrow A(t) = P_0^t(\gamma, \hat{\gamma})A_0 \in T^*M \otimes T\hat{M} \quad \forall t \in I,$$
$$A_0 \in Q \quad \Longrightarrow \quad A(t) = P_0^t(\gamma, \hat{\gamma})A_0 \in Q \quad \forall t \in I,$$

and

$$P_0^t(\gamma, \hat{\gamma})A_0 = P_0^t(\hat{\gamma}) \circ A_0 \circ P_t^0(\gamma) \quad \forall t \in I. \tag{13}$$

Let $T(M \times \hat{M}) \times_{M \times \hat{M}} (T^*(M) \otimes T(\hat{M}))$ be the total space of the product vector bundle $\pi_{T(M \times \hat{M})} \times_{M \times \hat{M}} \pi_{T^*(M) \otimes T(\hat{M})}$ over $M \times \hat{M}$. We will define certain *lift* operations corresponding to parallel translation of elements of $T^*M \otimes T\hat{M}$.

Definition 3.9 The *No-Spinning lift* is defined to be the map

$$\mathscr{L}_{\mathrm{NS}} : T(M \times \hat{M}) \times_{M \times \hat{M}} \big(T^*(M) \otimes T(\hat{M})\big) \to T\big(T^*(M) \otimes T(\hat{M})\big),$$

such that, if $q = (x, \hat{x}; A) \in T^*(M) \otimes T(\hat{M})$, $X \in T|_x M$, $\hat{X} \in T|_{\hat{x}} \hat{M}$ and $t \mapsto (\gamma(t), \hat{\gamma}(t))$ is a smooth curve on in $M \times \hat{M}$ defined on an open interval $I \ni 0$ s.t. $\dot{\gamma}(0) = X$, $\dot{\hat{\gamma}}(0) = \hat{X}$, then one has

$$\mathscr{L}_{\mathrm{NS}}((X, \hat{X}), q) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 P_0^t(\gamma, \hat{\gamma})A \in T|_q\big(T^*(M) \otimes T(\hat{M})\big). \tag{14}$$

The smoothness of the map $\mathscr{L}_{\mathrm{NS}}$ can be easily seen by using local trivializations. We will usually use a notation $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q$ for $\mathscr{L}_{\mathrm{NS}}(\overline{X}, q)$ when $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ and $q = (x, \hat{x}; A) \in T^*(M) \otimes T(\hat{M})$. In particular, when $\overline{X} \in \mathrm{VF}(M \times \hat{M})$, we get a *lifted vector field* on $T^*(M) \otimes T(\hat{M})$ given by $q \mapsto \mathscr{L}_{\mathrm{NS}}(\overline{X})|_q$. The smoothness of $\mathscr{L}_{\mathrm{NS}}(\overline{X})$ for $\overline{X} \in \mathrm{VF}(M \times \hat{M})$ follows immediately from the smoothness of the map $\mathscr{L}_{\mathrm{NS}}$. Notice that, by Proposition 3.8, the No-Spinning lift map $\mathscr{L}_{\mathrm{NS}}$ restricts to

$$\mathscr{L}_{\mathrm{NS}} : T(M \times \hat{M}) \times_{M \times \hat{M}} Q \to TQ,$$

where $T(M \times \hat{M}) \times_{M \times \hat{M}} Q$ is the total space of the fiber product $\pi_{T(M \times \hat{M})} \times_{M \times \hat{M}} \pi_Q$.

Definition 3.10 The *No-Spinning (NS) distribution* $\mathcal{D}_{\mathrm{NS}}$ on $T^*M \otimes T\hat{M}$ is a $2n$-dimensional smooth distribution defined pointwise by

$$\mathcal{D}_{\mathrm{NS}}|_q = \mathscr{L}_{\mathrm{NS}}(T|_{(x,\hat{x})}(M \times \hat{M}))|_q, \tag{15}$$

with $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$. Since $\mathcal{D}_{\mathrm{NS}}|_Q \subset TQ$ (by Proposition 3.8) this distribution restricts to a $2n$-dimensional smooth distribution on Q which we also denote by $\mathcal{D}_{\mathrm{NS}}$ (instead of $\mathcal{D}_{\mathrm{NS}}|_Q$).

The No-Spinning lift $\mathscr{L}_{\mathrm{NS}}$ will also be called $\mathcal{D}_{\mathrm{NS}}$-lift since it maps vectors of $M \times \hat{M}$ to vectors in $\mathcal{D}_{\mathrm{NS}}$. The distribution $\mathcal{D}_{\mathrm{NS}}$ is smooth since $\mathscr{L}_{\mathrm{NS}}(\overline{X})$ is smooth for any smooth vector field $\overline{X} \in \mathrm{VF}(M \times \hat{M})$. Also, the fact that the rank of $\mathcal{D}_{\mathrm{NS}}$ exactly is $2n$ follows from the next proposition, which itself follows immediately from Eq. (14).

Proposition 3.11 For every $q=(x,\hat{x};A)\in T^*M\otimes T\hat{M}$ and $\overline{X}\in T|_{(x,\hat{x})}(M\times\hat{M})$, one has

$$(\pi_{T^*M\otimes T\hat{M}})_*(\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q)=\overline{X},$$

and in particular $(\pi_Q)_*(\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q)=\overline{X}$ if $q\in Q$.

Thus $(\pi_{T^*M\otimes T\hat{M}})_*$ (resp. $(\pi_Q)_*$) maps $\mathcal{D}_{\mathrm{NS}}|_{(x,\hat{x};A)}$ isomorphically onto $T|_{(x,\hat{x})}(M\times\hat{M})$ for every $(x,\hat{x};A)\in T^*M\otimes T\hat{M}$ (resp. $(x,\hat{x};A)\in Q$) and the inverse map of $(\pi_{T^*M\otimes T\hat{M}})_*|_{\mathcal{D}_{\mathrm{NS}}|_q}$ (resp. $(\pi_Q)_*|_{\mathcal{D}_{\mathrm{NS}}|_q}$) is $\overline{X}\mapsto\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q$.

The following basic formula for the lift $\mathscr{L}_{\mathrm{NS}}$ will be useful.

Theorem 3.12 For $\overline{X}\in T|_{(x,\hat{x})}(M\times\hat{M})$ and $A\in\Gamma(\pi_{T^*M\otimes T\hat{M}})$, we have

$$\mathscr{L}_{\mathrm{NS}}(\overline{X})|_{A|_{(x,\hat{x})}}=A_*(\overline{X})-\nu\big(\overline{\nabla}_{\overline{X}}A\big)|_{A|_{(x,\hat{x})}},\tag{16}$$

where ν denotes the vertical derivative in the vector bundle $\pi_{T^*M\otimes T\hat{M}}$ and A_* is the map $T(M\times\hat{M})\to T(T^*M\otimes T\hat{M})$.

Proof. Choose smooth paths $\gamma:[-1,1]\to M$, $\hat{\gamma}:[-1,1]\to\hat{M}$ such that $(\dot{\gamma}(0),\dot{\hat{\gamma}}(0))=\overline{X}$ and take an arbitrary $f\in C^\infty(T^*M\otimes T\hat{M})$. Define $\tilde{A}(t)=P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})}$. Then

$$\mathscr{L}_{\mathrm{NS}}(\overline{X})|_{A|_{(x,\hat{x})}}=\dot{\tilde{A}}(0)=\tilde{A}_*(\frac{\partial}{\partial t}).$$

Also, it is known that (see e.g. [30], p.29)

$$P_t^0(\gamma,\hat{\gamma})(A|_{(\gamma(t),\hat{\gamma}(t))})=A|_{(x,\hat{x})}+t\overline{\nabla}_{\overline{X}}A+t^2F(t),\tag{17}$$

with $t\mapsto F(t)$ a C^∞-function $]-1,1[\to T^*|_xM\otimes T|_{\hat{x}}\hat{M}$. Moreover, one has

$$\begin{aligned}
&\big(A_*(\overline{X})-\tilde{A}_*(\frac{\partial}{\partial t})\big)f=\lim_{t\to0}\frac{f(A|_{(\gamma(t),\hat{\gamma}(t))})-f(P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})})}{t}\\
=&\lim_{t\to0}\frac{f(P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})}+tP_0^t(\gamma,\hat{\gamma})\overline{\nabla}_{\overline{X}}A+t^2P_0^t(\gamma,\hat{\gamma})F(t))-f(P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})})}{t}\\
=&\lim_{t\to0}\frac{1}{t}\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}f\big(P_0^t(\gamma,\hat{\gamma})A|_{(x,\hat{x})}+sP_0^t(\gamma,\hat{\gamma})\overline{\nabla}_{\overline{X}}A+s^2P_0^t(\gamma,\hat{\gamma})F(t)\big)\mathrm{d}s\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}f\big(A|_{(x,\hat{x})}+s\overline{\nabla}_{\overline{X}}A+s^2F(0)\big)=\nu(\overline{\nabla}_{\overline{X}}A)|_{A|_{(x,\hat{x})}}f.
\end{aligned}$$

□

We shall write Eq. (16) from now on with a compressed notation

$$\mathscr{L}_{\mathrm{NS}}(\overline{X})|_A=A_*(\overline{X})-\nu(\overline{\nabla}_{\overline{X}}A)|_A.$$

Remark 3.13 If $A\in\Gamma(\pi_{T^*(M)\otimes T(\hat{M})})$ and $q:=A|_{(x,\hat{x})}\in Q$ (e.g. if $A\in\Gamma(\pi_Q)$), then on the right hand side of (16), both terms are elements of $T|_q(T^*M\otimes T\hat{M})$ but their difference is actually an element of $T|_qQ$.

As a trivial corollary of the theorem, one gets the following.

Corollary 3.14 Suppose $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ is an a.c. curve on $T^*M \otimes T\hat{M}$ or Q defined on an open real interval I. Then, for a.e. $t \in I$,

$$\mathscr{L}_{\mathrm{NS}}\big(\dot{\gamma}(t), \dot{\hat{\gamma}}(t)\big)\big|_{q(t)} = \dot{A}(t) - \nu\big(\overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A\big)\big|_{q(t)}.$$

Remark 3.15 In [], we studied the control system associated to distribution $\mathcal{D}_{\mathrm{NS}}$ and this corresponds to rolling problem with no-spin condition (Eq. (9)) only. We are able to completely characterize the corresponding orbits in terms of the holonomy groups of (M, g), $(\hat{M}, \hat{g})$ and thus fully address the controllability issue.

3.2.2 The Rolling Distribution $\mathcal{D}_{\mathrm{R}}$

We next define a distribution which will correspond to the rolling with neither slipping nor spinning. As regards the rolling of one manifold onto another one, the admissible curve $q(\cdot)$ must verify the no-spinning condition (9) and no-slipping condition (11) that we recall next. Since $q(\cdot)$ is tangent to $\mathcal{D}_{\mathrm{NS}}$, we have $A(t) = P_0^t(x, \hat{x})A(0)$, and the no-slipping condition (11) writes $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$. It forces one to have, for a.e. t,

$$\dot{q}(t) = \mathscr{L}_{\mathrm{NS}}\big(\dot{\gamma}(t), A(t)\dot{\gamma}(t)\big)\big|_{q(t)}.$$

Evaluating at $t = 0$ and noticing that if $q_0 := q(0)$, with $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and $\dot{\gamma}(0) =: X \in T|_{x_0}M$ are arbitrary, we get

$$\dot{q}(0) = \mathscr{L}_{\mathrm{NS}}(X, A_0X)|_{q_0}.$$

This motivates the following definition.

Definition 3.16 For $q = (x, \hat{x}; A) \in Q$, we define the *Rolling lift* or $\mathcal{D}_{\mathrm{R}}$-lift as a bijective linear map

$$\mathscr{L}_{\mathrm{R}} : T|_xM \times Q|_{(x,\hat{x})} \to T|_qQ,$$

given by

$$\mathscr{L}_{\mathrm{R}}(X, q) = \mathscr{L}_{\mathrm{NS}}(X, AX)|_q. \tag{18}$$

This map naturally induces $\mathscr{L}_{\mathrm{R}} : \mathrm{VF}(M) \to \mathrm{VF}(Q)$ as follows. For $X \in \mathrm{VF}(M)$ we define $\mathscr{L}_{\mathrm{R}}(X)$, the *Rolling lifted* vector field associated to X, by

$$\begin{aligned} \mathscr{L}_{\mathrm{R}}(X) : Q &\to T(Q), \\ q &\mapsto \mathscr{L}_{\mathrm{R}}(X)|_q, \end{aligned}$$

where $\mathscr{L}_{\mathrm{R}}(X)|_q := \mathscr{L}_{\mathrm{R}}(X, q)$.

The Rolling lift map $\mathscr{L}_{\mathrm{R}}$ allows one to construct a distribution on Q (see [7]) reflecting both of the rolling restrictions of motion defined by the no-spinning condition, Eq. (9), and the no-slipping condition, Eq. (11).

Definition 3.17 The *rolling distribution* $\mathcal{D}_{\mathrm{R}}$ on Q is the n-dimensional smooth distribution defined pointwise by

$$\mathcal{D}_{\mathrm{R}}|_q = \mathscr{L}_{\mathrm{R}}(T|_xM)|_q, \tag{19}$$

for $q = (x, \hat{x}; A) \in Q$.

The Rolling lift $\mathscr{L}_{\mathrm{R}}$ will also be called $\mathcal{D}_{\mathrm{R}}$-lift since it maps vectors of M to vectors in $\mathcal{D}_{\mathrm{R}}$. Thus an absolutely continuous curve $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in Q is a rolling curve if and only if it is a.e. tangent to $\mathcal{D}_{\mathrm{R}}$ i.e., $\dot{q}(t) \in \mathcal{D}_{\mathrm{R}}|_{q(t)}$ for a.e. t or, equivalently, if $\dot{q}(t) = \mathscr{L}_{\mathrm{R}}(\dot{\gamma}(t))|_{q(t)}$ for a.e. t.

Define $\pi_{Q,M} = \mathrm{pr}_1 \circ \pi_Q : Q \to M$ and notice that its differential $(\pi_{Q,M})_*$ maps each $\mathcal{D}_{\mathrm{R}}|_{(x,\hat{x};A)}$, $(x, \hat{x}; A) \in Q$, isomorphically onto $T|_x M$. Similarly one defines $\pi_{Q,\hat{M}} = \mathrm{pr}_2 \circ \pi_Q : Q \to \hat{M}$.

Proposition 3.18 For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and absolutely continuous $\gamma : [0, a] \to M$, $a > 0$, such that $\gamma(0) = x_0$, there exists a unique absolutely continuous $q : [0, a'] \to Q$, $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, with $0 < a' \le a$ (and a' maximal with the latter property), which is tangent to $\mathcal{D}_{\mathrm{R}}$ a.e. and $q(0) = q_0$. We denote this unique curve q by

$$t \mapsto q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = (\gamma(t), \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t); A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)),$$

and refer to it as the *rolling curve along* γ *with initial position* q_0. In the case that $\hat{M}$ is a complete manifold one has $a' = a$.

Conversely, any absolutely continuous curve $q : [0, a] \to Q$, which is a.e. tangent to $\mathcal{D}_{\mathrm{R}}$, is a rolling curve along $\gamma = \pi_{Q,M} \circ q$ i.e., has the form $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q(0))$.

Proof. We need to show only that completeness of $(\hat{M}, \hat{g})$ implies that $a' = a$. In fact, $\hat{X}(t) := A_0 \int_0^t P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s$ defines an a.c. curve $t \mapsto \hat{X}(t)$ in $T|_{\hat{x}_0}\hat{M}$ defined on $[0, a]$ and the completeness of $\hat{M}$ implies that there is a unique a.c. curve $\hat{\gamma}$ on $\hat{M}$ defined on $[0, a]$ such that $\hat{X}(t) = \int_0^t P_s^0(\hat{\gamma})\dot{\hat{\gamma}}(s)\mathrm{d}s$ for all $t \in [0, a]$. Defining $A(t) = P_0^t(\hat{\gamma}) \circ A_0 \circ P_t^0(\gamma)$, $t \in [0, a]$ we notice that $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ is the rolling curve along γ starting at q_0 that is defined on the interval $[0, a]$. Hence $a' = a$.

□

Of course, it is not important in the previous result that we start the parametrization of the curve γ at $t = 0$.

Remark 3.19 It follows immediately from the uniqueness statement of the previous theorem that, if $\gamma : [a, b] \to M$ and $\omega : [c, d] \to M$ are two a.c. curves with $\gamma(b) = \omega(c)$ and $q_0 \in Q$, then

$$q_{\mathcal{D}_{\mathrm{R}}}(\omega \sqcup \gamma, q_0) = q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(b)) \sqcup q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0). \tag{20}$$

On the group $\Omega_{x_0}(M)$ of piecewise differentiable loops of M based at x_0 one has

$$q_{\mathcal{D}_{\mathrm{R}}}(\omega.\gamma, q_0) = q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1)).q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0),$$

where $\gamma, \omega \in \Omega_{x_0}(M)$.

In the case where the curve γ on M is a geodesic, we can give a more precise form of the rolling curve along γ with a given initial position.

Proposition 3.20 Consider $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, $X \in T|_{x_0}M$ and $\gamma : [0, a] \to M$; $\gamma(t) = \exp_{x_0}(tX)$, a geodesic of (M, g) with $\gamma(0) = x_0$, $\dot{\gamma}(0) = X$. Then the rolling curve $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0) = (\gamma, \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0); A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)) : [0, a'] \to Q$, $0 < a' \le a$, along γ with initial position q_0 is given by

$$\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = \widehat{\exp}_{\hat{x}_0}(tA_0X), \quad A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = P_0^t(\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)) \circ A_0 \circ P_t^0(\gamma).$$

Of course, $a' = a$ if $\hat{M}$ is complete.

Proof. Let $0 < a' \le a$ such that $\hat{\gamma}(t) := \widehat{\exp}_{\hat{x}_0}(tA_0X)$ is defined on $[0, a']$. Then, by proposition 3.8, $q(t) := (\gamma(t), \hat{\gamma}(t); A(t))$ with $A(t) := P_0^t(\hat{\gamma}) \circ A_0 \circ P_t^0(\gamma)$, $t \in [0, a']$, is a curve on Q and $A(t)$ is parallel to $(\gamma, \hat{\gamma})$ in $M \times \hat{M}$. Therefore $t \mapsto q(t)$ is tangent to $\mathcal{D}_{\mathrm{NS}}$ on $[0, a']$ and thus $\dot{q}(t) = \mathscr{L}_{\mathrm{NS}}(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))|_{q(t)}$. Moreover, since γ and $\hat{\gamma}$ are geodesics,

$$A(t)\dot{\gamma}(t) = (P_0^t(\hat{\gamma}) \circ A_0)(P_t^0(\gamma)\dot{\gamma}(t)) = P_0^t(\hat{\gamma})(A_0X) = \dot{\hat{\gamma}}(t),$$

which shows that for $t \in [0, a']$,

$$\begin{aligned} \dot{q}(t) &= \mathscr{L}_{\mathrm{NS}}(\dot{\gamma}(t), A(t)\dot{\gamma}(t))|_{q(t)} \\ &= \mathscr{L}_{\mathrm{R}}(\dot{\gamma}(t))\big|_{q(t)}. \end{aligned}$$

Hence $t \mapsto q(t)$ is tangent to $\mathcal{D}_{\mathrm{R}}$ i.e., it is a rolling curve along γ with initial position $q(0) = (\gamma(0), \hat{\gamma}(0); A(0)) = (x_0, \hat{x}_0; A_0) = q_0$.

□

Remark 3.21 If $\gamma(t) = \exp_{x_0}(tA_0X)$ and $q_0 = (x_0, \hat{x}_0; A_0)$, the statement of the proposition can be written in a compact form as

$$A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = P_0^t\big(s \mapsto \overline{\exp}_{(x_0, \hat{x}_0)}(s(X, A_0X))\big)A_0,$$

for all t where defined.

The next proposition describes the symmetry of the study of the rolling problem of (M, g) rolling against $(\hat{M}, \hat{g})$ to the problem of $(\hat{M}, \hat{g})$ rolling against (M, g).

Proposition 3.22 Let $\widehat{\mathcal{D}_{\mathrm{R}}}$ be the rolling distribution in $\hat{Q} := Q(\hat{M}, M)$. Then the map

$$\iota : Q \to \hat{Q}; \quad \iota(x, \hat{x}; A) = (\hat{x}, x; A^{-1})$$

is a diffeomorphism of Q onto $\hat{Q}$ and

$$\iota_* \mathcal{D}_{\mathrm{R}} = \widehat{\mathcal{D}_{\mathrm{R}}}.$$

In particular, $\iota(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)) = \mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\iota(q))$.

Proof. It is obvious that ι is a diffeomorphism (with the obvious inverse map) and for an a.c. path $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in Q, $(\iota \circ q)(t) = (\hat{\gamma}(t), \gamma(t); A(t)^{-1})$ is a.c. in $\hat{Q}$ and for a.e. t,

$$\begin{cases} \dot{\hat{\gamma}}(t) = A(t)\dot{\gamma}(t) \\ A(t) = P_0^t(\hat{\gamma}) \circ A(0) \circ P_t^0(\gamma) \end{cases} \iff \begin{cases} \dot{\gamma}(t) = A(t)^{-1}\dot{\hat{\gamma}}(t) \\ A(t)^{-1} = P_0^t(\gamma) \circ A(0)^{-1} \circ P_t^0(\hat{\gamma}) \end{cases}.$$

These simple remarks prove the claims.

□

Remark 3.23 Notice that Definitions 3.16 and 3.17 make sense not only in Q but also in the space $T^*M \otimes T\hat{M}$. It is easily seen that $\mathcal{D}_{\mathrm{R}}$ defined on $T^*M \otimes T\hat{M}$ by Eq. (19) is actually tangent to Q so its restriction to Q gives exactly $\mathcal{D}_{\mathrm{R}}$ on Q as defined above. Similarly, Propositions 3.18, 3.20 and 3.22 still hold if we replace Q by $T^*M \otimes T\hat{M}$ and $\hat{Q}$ by $T^*\hat{M} \otimes TM$ everywhere in their statements.

3.3 Lie brackets of vector fields on Q

In this section, we compute commutators of the vectors fields of $T^*M \otimes T\hat{M}$ and Q with respect to the splitting of $T(T^*M \otimes T\hat{M})$ (resp. TQ) as a direct sum $\mathcal{D}_{\mathrm{NS}} \oplus V(\pi_{T^*M\otimes T\hat{M}})$ (resp. $\mathcal{D}_{\mathrm{NS}} \oplus V(\pi_Q)$). The main results are Propositions 3.35, 3.35 and 3.37. These computations will serve as preliminaries for the Lie bracket computations relative to the rolling distribution $\mathcal{D}_{\mathrm{R}}$ studied in the next section. It is convenient to make computations in $T^*M \otimes T\hat{M}$ and then to restrict the results to Q.

3.3.1 Computational tools

The next lemmas will be useful in the subsequent calculations.

Lemma 3.24 Let $(x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ (resp. $(x, \hat{x}; A) \in Q$). Then there exists a local $\pi_{T^*M\otimes T\hat{M}}$-section (resp. π_Q-section) $\tilde{A}$ around $(x, \hat{x})$ such that $\tilde{A}|_{(x,\hat{x})} = A$ and $\overline{\nabla}_{\overline{X}}\tilde{A} = 0$ for all $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$.

Proof. Let U be an open neighborhood of the origin of $T|_{(x,\hat{x})}(M \times \hat{M})$, where the $\overline{g}$-exponential map $\overline{\exp} : U \to M \times \hat{M}$ is a diffeomorphism onto its image. Parallel translate A along geodesics $t \mapsto \overline{\exp}(t\overline{X})$, $\overline{X} \in U$, to get a local section $\tilde{A}$ of $T^*(M) \otimes T(\hat{M})$ in a neighborhood of $\overline{x} = (x, \hat{x})$. More explicitly, one has

$$\tilde{A}|_{\overline{y}} = P_0^1\big(t \mapsto \overline{\exp}\big(t(\overline{\exp}_{\overline{x}})^{-1}(\overline{y})\big)\big)A,$$

for $\overline{y} \in U$. If $(x, \hat{x}; A) \in Q$, this actually provides a local π_Q-section. Moreover, we clearly have $\overline{\nabla}_{\overline{X}}\tilde{A} = 0$ for all $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$.

□

Notice that the choice of $\tilde{A}$ corresponding to $(x, \hat{x}; A)$ is, of course, not unique. The proof of the following lemma is obvious and hence omitted.

Lemma 3.25 Let $\tilde{A}$ be a smooth local $\pi_{T^*M\otimes T\hat{M}}$-section and $\tilde{A}|_{(x,\hat{x})} = A$. Then, for any vector fields $\overline{X}, \overline{Y} \in \mathrm{VF}(M \times \hat{M})$ such that $\overline{X}|_{(x,\hat{x})} = (X, \hat{X})$, $\overline{Y}|_{(x,\hat{x})} = (Y, \hat{Y})$, one has

$$([\overline{\nabla}_{\overline{X}}, \overline{\nabla}_{\overline{Y}}]\tilde{A})|_{(x,\hat{x})} = -AR(X, Y) + \hat{R}(\hat{X}, \hat{Y})A + (\overline{\nabla}_{[\overline{X},\overline{Y}]}\tilde{A})|_{(x,\hat{x})}. \tag{21}$$

Here $[\overline{\nabla}_{\overline{X}}, \overline{\nabla}_{\overline{Y}}]$ is given by $\overline{\nabla}_{\overline{X}} \circ \overline{\nabla}_{\overline{Y}} - \overline{\nabla}_{\overline{Y}} \circ \overline{\nabla}_{\overline{X}}$ and is an $\mathbb{R}$-linear map on the set of local sections of $\pi_{T^*M\otimes T\hat{M}}$ around $(x, \hat{x})$.

We next define the actions of vectors $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q \in T|_q(T^*M \otimes T\hat{M})$, $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$, and $\nu(B)|_q \in V|_q(\pi_{T^*M\otimes T\hat{M}})$, $B \in T|_x^*M \otimes T|_{\hat{x}}\hat{M}$, on certain bundle maps instead of just functions (e.g. from $C^\infty(T^*M \otimes T\hat{M})$). Recall that if

$\eta : E \to N$ is a vector bundle and $y \in N$, $u \in E|_y = \eta^{-1}(y)$, we have defined the isomorphism

$$\nu_\eta|_u : E|_y \to V|_u(\eta); \quad \nu_\eta|_u(v)(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(u+tv), \quad \forall f \in C^\infty(E).$$

We normally omit the index η in ν_η, when it is clear from the context, and simply write ν instead of ν_η and it is sometimes more convenient to write $\nu(v)|_u$ for $\nu|_u(v)$. By using this we make the following definition.

Definition 3.26 Suppose B is a smooth manifold, $\eta : E \to N$ a vector bundle, $\tau : B \to N$ and $F : B \to E$ smooth maps such that $\eta \circ F = \tau$. Then, for $b \in B$ and $\mathcal{V} \in V|_b(\tau)$, we define the vertical derivative of F as

$$\mathcal{V}F := \nu|_{F(b)}^{-1}(F_*\mathcal{V}) \in E|_{\tau(b)}.$$

This is well defined since $F_*\mathcal{V} \in V|_{F(b)}(\eta)$. In this matter, we will show the following simple lemma that will be used later on.

Lemma 3.27 Let N be a smooth manifold, $\eta : E \to N$ a vector bundle, $\tau : B \to N$ a smooth map, $\mathcal{O} \subset B$ an immersed submanifold and $F : \mathcal{O} \to E$ a smooth map such that $\eta \circ F = \tau|_{\mathcal{O}}$.

(i) For every $b_0 \in \mathcal{O}$, there exists an open neighbourhood V of b_0 in $\mathcal{O}$, an open neighbourhood $\tilde{V}$ of b_0 in B s. t. $V \subset \tilde{V}$ and a smooth map $\tilde{F} : \tilde{V} \to E$ such that $\eta \circ \tilde{F} = \tau|_{\tilde{V}}$ and $\tilde{F}|_V = F|_V$. We call $\tilde{F}$ a local extension of F around b_0.

(ii) Suppose $\tau : B \to N$ is also a vector bundle and $\tilde{F}$ is any local extension of F around b_0 as in case (i). Then if $v \in B|_{\tau(b_0)}$ is such that $\nu|_{b_0}(v) \in T|_{b_0}\mathcal{O}$, one has

$$\nu|_{b_0}(v)(F) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \tilde{F}(b_0 + tv) \in E|_{\tau(b_0)},$$

where on the right hand side one views $t \mapsto \tilde{F}(b_0 + tv)$ as a map into a fixed (i.e. independent of t) vector space $E|_{F(b_0)}$ and the derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ is just the classical derivative of a vector valued map (and not a tangent vector).

Proof. (i) For a given $b_0 \in \mathcal{O}$, take a neighbourhood W of $y_0 := \tau(b_0)$ in N such that there exists a local frame $v_1, \ldots, v_k$ of η defined on W (here $k = \dim E - \dim N$). Since $\eta \circ F = \tau|_{\mathcal{O}}$, it follows that

$$F(b) = \sum_{i=1}^{k} f_i(b) v_i|_{\tau(b)}, \quad \forall b \in \tau^{-1}(W) \cap \mathcal{O},$$

for some smooth functions $f_i : \tau^{-1}(W) \cap \mathcal{O} \to \mathbb{R}$, $i = 1, \ldots, k$. Now one can choose a small open neighbourhood V of b_0 in $\mathcal{O}$ and an open neigbourhood $\tilde{V}$ of b_0 in B such that $V \subset \tilde{V} \subset \tau^{-1}(W)$ and there exist smooth $\tilde{f}_1, \ldots, \tilde{f}_k : \tilde{V} \to \mathbb{R}$ extending the functions $f_i|_V$ i.e. $\tilde{f}_i|_V = f_i|_V$, $i = 1, \ldots, k$. To finish the proof of case (i), it suffices to define $\tilde{F} : \tilde{V} \to E$ by

$$\tilde{F}(b) = \sum_{i=1}^{k} \tilde{f}_i(b) v_i|_{\tau(b)}, \quad \forall b \in \tilde{V}.$$

(ii) The fact that $t \mapsto \tilde{F}(b_0 + tv)$ is a map into a fixed vector space $E|_{F(b_0)}$ is clear since $\tilde{F}(b_0 + tv) \in E|_{\eta(\tilde{F}(b_0+tv))} = E|_{\tau(b_0+tv)} = E|_{\tau(b_0)}$. Since $F|_V = \tilde{F}|_V$ and $\nu|_{b_0}(v) \in T|_{b_0}V$, we have $F_*\nu|_{b_0}(v) = \tilde{F}_*\nu|_{b_0}(v)$. Also, $t \mapsto b_0 + tv$ is a curve in $E|_{\tau(b_0)}$, and hence in E, whose tangent vector at $t = 0$ is exactly $\nu|_{b_0}(v)$. Hence

$$\nu|_{F(b_0)}(\nu|_{b_0}(v)F) = F_*\nu|_{b_0}(v) = \tilde{F}_*\nu|_{b_0}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \tilde{F}(b_0 + tv).$$

Here on the rightmost side, the derivative $=: T$ is still viewed as a tangent vector of E at $\tilde{F}(b_0)$ i.e. $t \mapsto \tilde{F}(b_0 + tv)$ is thought of as a map into E. On the other hand, if one views $t \mapsto \tilde{F}(b_0 + tv)$ as a map into a fixed linear space $E|_{\tau(b_0)}$, its derivative $=: D$ at $t = 0$, as the usual derivative of vector valued maps, is just $D = \nu|_{F(b_0)}^{-1}(T)$. In the statement, it is exactly D whose expression we wrote as $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \tilde{F}(b_0 + tv)$. This completes the proof. $\square$

Remark 3.28 The advantage of the formula in case (ii) of the above lemma is that it simplifies in many cases the computations of τ-vertical derivatives because $t \mapsto \tilde{F}(b_0+tv)$ is a map from a real interval into a *fixed* vector space $E|_{F(b_0)}$ and hence we may use certain computational tools (e.g. Leibniz rule) coming from the ordinary vector calculus.

Let $\mathcal{O}$ be an immersed submanifold of $T^*M \otimes T\hat{M}$ and write $\pi_{\mathcal{O}} := \pi_{T^*M\otimes T\hat{M}}|_{\mathcal{O}}$. Then if $\overline{T} : \mathcal{O} \to T^k_m(M \times \hat{M})$ with $\pi_{T^k_m(M\times\hat{M})} \circ \overline{T} = \pi_{\mathcal{O}}$ (i.e. $\overline{T} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^k_m(M\times\hat{M})})$) and if $q = (x, \hat{x}; A) \in \mathcal{O}$ and $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ are such that $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q \in T|_q\mathcal{O}$, we next want to define what it means to take the derivative $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q\overline{T}$. Our main interest will be the case where $k = 0$, $m = 1$ i.e. $T^k_m(M \times \hat{M}) = T(M \times \hat{M})$, but some arguments below require this slightly more general setting.

First, for a moment, we take $\mathcal{O} = T^*M \otimes T\hat{M}$. Choose some local $\pi_{T^*M\otimes T\hat{M}}$-section $\tilde{A}$ defined on a neighbourhood of $(x, \hat{x})$ such that $\tilde{A}|_{(x,\hat{x})} = A$ and define

$$\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q\overline{T} := \overline{\nabla}_{\overline{X}}(\overline{T}(\tilde{A})) - \nu(\overline{\nabla}_{\overline{X}}\tilde{A})|_q\overline{T} \in T^k_m|_{(x,\hat{x})}(M \times \hat{M}), \tag{22}$$

which is inspired by Eq. (16). Here as usual, $\tilde{T}(\tilde{A}) = \tilde{T} \circ \tilde{A}$ is a locally defined (k, m)-tensor field on $M \times \hat{M}$. Note that this does not depend on the choice of $\tilde{A}$ since if $\overline{\omega} \in \Gamma(\pi_{T^m_k(M\times\hat{M})})$ and if we write $(\overline{T}\overline{\omega})(q) := \overline{T}(q)\overline{\omega}|_{(x,\hat{x})}$ as a full contraction for $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, whence $\overline{T}\overline{\omega} \in C^\infty(T^*M \otimes T\hat{M})$, we may compute (where all the contractions are full)

$$\begin{aligned}
(\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q\overline{T})\overline{\omega} =& \big(\overline{\nabla}_{\overline{X}}(\overline{T}(\tilde{A}))\big)\overline{\omega} - \big(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{T}(A + t\overline{\nabla}_{\overline{X}}\tilde{A})\big)\overline{\omega} \\
=& \overline{\nabla}_{\overline{X}}(\overline{T}(\tilde{A})\overline{\omega}) - \overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega} - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \big(\overline{T}(A + t\overline{\nabla}_{\overline{X}}\tilde{A})\overline{\omega}\big) \\
=& \overline{\nabla}_{\overline{X}}((\overline{T}\overline{\omega}))(\tilde{A})\big) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 (\overline{T}\overline{\omega})(A + t\overline{\nabla}_{\overline{X}}\tilde{A}) - \overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega}
\end{aligned}$$

i.e.

$$(\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q\overline{T})\overline{\omega} = \mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(\overline{T}\overline{\omega}) - \overline{T}(q)\overline{\nabla}_{\overline{X}}\overline{\omega}, \tag{23}$$

for all $\overline{\omega} \in \Gamma(\pi_{T_k^m(M\times\hat{M})})$ and where $\mathscr{L}_{\rm NS}(\overline{X})|_q$ on the right hand side acts as a tangent vector to a function $\overline{T}\overline{\omega} \in C^\infty(T^*M \otimes T\hat{M})$ as defined previously.

The right hand side is independent of any choice of local extension $\tilde{A}$ of A (i.e. $\tilde{A}|_{(x,\hat{x})} = A$), it follows that the definition of $\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T}$ is independent of this choice as well. Now if $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is just an immersed submanifold, we take the formula (23) as the definition of $\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T}$.

Definition 3.29 Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold and $q = (x,\hat{x};A) \in \mathcal{O}$, $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ be such that $\mathscr{L}_{\rm NS}(\overline{X})|_q \in T|_q\mathcal{O}$. Then for $\overline{T} : \mathcal{O} \to T_m^k(M \times \hat{M})$ such that $\pi_{T_m^k(M\times\hat{M})} \circ \overline{T} = \pi_{\mathcal{O}}$, we define $\mathscr{L}_{\rm NS}(\overline{X})|_q\overline{T}$ to be the unique element in $T_m^k|_{(x,\hat{x})}(M \times \hat{M})$ such that Eq. (23) holds for every $\overline{\omega} \in \Gamma(\pi_{T_k^m(M\times\hat{M})})$, and call it the derivative of $\overline{T}$ with respect to $\mathscr{L}_{\rm NS}(\overline{X})|_q$.

We now to provide the (unique) decomposition of any vector field of $T^*M \otimes T\hat{M}$ defined over $\mathcal{O}$ (not necessarily tangent to it) according to the decomposition $T(T^*M \otimes T\hat{M}) = \mathcal{D}_{\rm NS} \oplus V(\pi_{T^*M\otimes T\hat{M}})$.

Proposition 3.30 Let $\mathcal{X} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(T^*M\otimes T\hat{M})})$ be a smooth bundle map (i.e. a vector field of $T^*M \otimes T\hat{M}$ along $\mathcal{O}$) where $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is a smooth immersed submanifold. Then there are unique smooth bundle maps $\overline{T} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$, $U \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$ such that

$$\mathcal{X}|_q = \mathscr{L}_{\rm NS}(\overline{T}(q))|_q + \nu(U(q))|_q, \quad q \in \mathcal{O}. \tag{24}$$

Proof. First of all, there are unique smooth vector fields

$$\mathcal{X}^h, \mathcal{X}^v \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(T^*M\otimes T\hat{M})}),$$

of $T^*M \otimes T\hat{M}$ along $\mathcal{O}$ such that

$$\mathcal{X}^h|_q \in \mathcal{D}_{\rm NS}|_q, \ \mathcal{X}^v|_q \in V|_q(\pi_{T^*M\otimes T\hat{M}}),$$

for all $q \in \mathcal{O}$ and $\mathcal{X} = \mathcal{X}^h + \mathcal{X}^v$. Then, we define

$$\overline{T}(q) = (\pi_{T^*M\otimes T\hat{M}})_*\mathcal{X}^h|_q, \quad U(q) = \nu|_q^{-1}(\mathcal{X}^v|_q),$$

where $q = (x,\hat{x};A) \in \mathcal{O}$ and $\nu|_q$ is the isomorphism

$$T^*|_xM \otimes T|_{\hat{x}}\hat{M} \to V|_q(\pi_{T^*M\otimes T\hat{M}}); \quad B \mapsto \nu(B)|_q.$$

This clearly proves the claims. □

Remark 3.31 The previous results shows that to know how to compute the Lie brackets of two vector fields $\mathcal{X}, \mathcal{Y} \in \mathrm{VF}(\mathcal{O})$ where $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is an immersed submanifold (e.g. $\mathcal{O} = Q$), one needs, in practice, just to know how to compute the Lie brackets between vectors fields of the form $q \mapsto \mathscr{L}_{\rm NS}(\overline{T}(q))|_q, \mathscr{L}_{\rm NS}(\overline{S}(q))$ and $q \mapsto \nu(U(q))|_q, \nu(V(q))|_q$ where $\mathcal{X}|_q = \mathscr{L}_{\rm NS}(\overline{T}(q))|_q + \nu(U(q))|_q$ and $\mathcal{Y}|_q = \mathscr{L}_{\rm NS}(\overline{S}(q))|_q + \nu(V(q))|_q$ as above.

Remark 3.32 Notice that if $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is an immersed submanifold, $q = (x, \hat{x}; A) \in \mathcal{O}$, $\mathcal{X} \in T|_q\mathcal{O}$ and $\overline{T} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^k_m(M\times\hat{M})})$, then we may define the derivative $\mathcal{X}\overline{T} \in T^k_m(M \times \hat{M})$ by decomposing $\mathcal{X} = \mathscr{L}_{\mathrm{NS}}(\overline{X})|_q + \nu(U)|_q$ for the unique $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ and $U \in (T^*M \otimes T\hat{M})|_{(x,\hat{x})}$.

We finish this subsection with some obvious but useful rules of calculation, that will be useful in the computations of Lie brackets on $\mathcal{O} \subset T^*M \otimes T\hat{M}$ and we will make use of them especially in section 5.

Lemma 3.33 Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $q = (x, \hat{x}; A) \in \mathcal{O}$, $\overline{T} \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^k_m(M\times\hat{M})})$, $F \in C^\infty(\mathcal{O})$, $h \in C^\infty(\mathbb{R})$, $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ such that $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q \in T|_q\mathcal{O}$ and finally $U \in (T^*M \times T\hat{M})|_{(x,\hat{x})}$ such that $\nu(U)|_q \in T|_q\mathcal{O}$ Then

(i) $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(F\overline{T}) = (\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q F)\overline{T}(q) + F(q)\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q\overline{T}$,

(ii) $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(h \circ F) = h'(F(q))\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q F$,

(iii) $\nu(U)|_q(F\overline{T}) = (\nu(U)|_q F)\overline{T}(q) + F(q)\nu(U)|_q\overline{T}$,

(iv) $\nu(U)|_q(h \circ F) = h'(F(q))\nu(U)|_q F$.

If $T : \mathcal{O} \to TM \subset T(M \times \hat{M})$ such that $T(q) \in T|_x M$ for all $q = (x, \hat{x}; A) \in \mathcal{O}$ and one writes (see Remark 3.34 below)

$$(\cdot)T(\cdot) : \mathcal{O} \to T\hat{M} \subset T(M \times \hat{M}); \quad q = (x, \hat{x}; A) \mapsto AT(q),$$

then

(v) $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q\big((\cdot)T(\cdot)\big) = A\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q T \in T|_{\hat{x}}\hat{M}$,

(vi) $\nu(U)|_q\big((\cdot)T(\cdot)\big) = UT(q) + A\nu(U)|_q T \in T|_{\hat{x}}\hat{M}$,

where $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q T, \nu(U)|_q T \in T|_x M$. Finally, if $Y \in \mathrm{VF}(M)$ is considered as a map $\mathcal{O} \to TM$; $(x', \hat{x}'; A') \mapsto Y|_{x'}$ and if we write $\overline{X} = (X, \hat{X}) \in T|_x M \oplus T|_{\hat{x}}\hat{M}$, then

(vii) $\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q Y = \nabla_X Y$.

Remark 3.34 In the cases (v) and (vii) we think of $T : \mathcal{O} \to TM$, to adapt to our previous notations, as a map $T : \mathcal{O} \to \mathrm{pr}_1^*(TM)$ where $\mathrm{pr}_1 : M \times \hat{M} \to M$ is the projection onto the first factor. Here $\mathrm{pr}_1^*(\pi_{TM})$ is a vector subbundle of $\pi_{T(M\times\hat{M})}$ which we wrote, slightly imprecisely, as $TM \subset T(M \times \hat{M})$ in the statement of the proposition. Thus $T(q') \in T|_{x'}M$ for all $q' = (x', \hat{x}'; A') \in \mathcal{O}$ just means that $\mathrm{pr}_1^*(\pi_{TM}) \circ T = \pi_{\mathcal{O}}$.

Proof. Items (i)-(iv) are immediate to derive. We next turn to an argument for the others. We take a small open neighbourhood V of q in $\mathcal{O}$, a small open neighbourhood $\tilde{V}$ of q in $T^*M \otimes T\hat{M}$ such that $V \subset \tilde{V}$ a smooth $\tilde{T} : \tilde{V} \to TM$ such that $\tilde{T}|_V = T|_V$ and $\tilde{T}(q') \in T|_{x'}M$ for all $q' = (x', \hat{x}'; A') \in \tilde{V}$. Such an extension $\tilde{T}$ of T is provided by Lemma 3.27 by taking $b_0 = q$, $\tau = \pi_{T^*M\otimes T\hat{M}}$, $\eta = \mathrm{pr}_1^*(\pi_{TM})$ with $\mathrm{pr}_1 : M \times \hat{M} = M$ the projection onto the first factor (see also Remark 3.34 above).

Then taking $t \mapsto \Gamma(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ to be any curve in $\mathcal{O}$ with $\Gamma(0) = q$, $\dot{\Gamma}(0) = \mathscr{L}_{\mathrm{NS}}(\overline{X})|_q$, we have

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q((\cdot)T(\cdot)) = \mathscr{L}_{\mathrm{NS}}(\overline{X})|_q((\cdot)\tilde{T}(\cdot)) \\
=&\overline{\nabla}_{\overline{X}}(A(\cdot)\tilde{T}(A(\cdot))) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 (A + t\overline{\nabla}_{\overline{X}}A(\cdot))\tilde{T}(A + t\overline{\nabla}_{\overline{X}}A(\cdot)) \\
=&(\overline{\nabla}_{\overline{X}}A(\cdot))\tilde{T}(q) + A\overline{\nabla}_{\overline{X}}(\tilde{T}(A(\cdot))) - (\overline{\nabla}_{\overline{X}}A(\cdot))\tilde{T}(q) - A\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \tilde{T}(A + t\overline{\nabla}_{\overline{X}}A(\cdot)) \\
=&A\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q\tilde{T} = A\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q T,
\end{aligned}
$$

where the first and the last steps follow from the facts that $((\cdot)\tilde{T}(\cdot))|_V = ((\cdot)T(\cdot))|_V$ and $\tilde{T}|_V = T|_V$. This gives (v).

To prove (vi) we compute

$$
\begin{aligned}
&\nu(U)|_q\big((\cdot)T(\cdot)\big) = \nu(U)|_q\big((\cdot)\tilde{T}(\cdot)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 (A + tU)\tilde{T}(A + tU) \\
=&\big(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 (A + tU)\big)\tilde{T}(q) + A\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \tilde{T}(A + tU) = UT(q) + A\nu(U)|_q\tilde{T} \\
=&UT(q) + A\nu(U)|_q T.
\end{aligned}
$$

Finally, we prove (vii). Suppose that $Y \in \mathrm{VF}(M)$. Then the map $\mathcal{O} \to TM$; $(x', \hat{x}'; A') \mapsto Y|_{x'}$ is nothing more than $Y \circ \mathrm{pr}_1 \circ \pi_{\mathcal{O}}$ where $\mathrm{pr}_1 : M \times \hat{M} \to M$ is the projection onto the first factor. Take a local $\pi_{T^*M\otimes T\hat{M}}$-section $\tilde{A}$ with $\tilde{A}|_{(x,\hat{x})} = A$. Then since $Y \circ \mathrm{pr}_1 \circ \pi_{\mathcal{O}} = Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}}|_{\mathcal{O}}$, we have

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(Y \circ \mathrm{pr}_1 \circ \pi_{\mathcal{O}}) = \mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}}) \\
=&\overline{\nabla}_{(X,\hat{X})}(Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}} \circ \tilde{A}) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 (Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}})(A + t\overline{\nabla}_{\overline{X}}\tilde{A}).
\end{aligned}
$$

But $(Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}} \circ \tilde{A})|_{(x',\hat{x}')} = Y|_{x'} = (Y, 0)|_{(x,\hat{x})}$ for all $(x', \hat{x}')$ and $(Y \circ \mathrm{pr}_1 \circ \pi_{T^*M\otimes T\hat{M}})(A + t\overline{\nabla}_{\overline{X}}\tilde{A}) = Y|_x$ for all t and hence

$$
\mathscr{L}_{\mathrm{NS}}(\overline{X})|_q(Y \circ \mathrm{pr}_1 \circ \pi_{\mathcal{O}}) = \overline{\nabla}_{(X,\hat{X})}(Y, 0) - 0 = \nabla_X Y.
$$

□

3.3.2 Computation of Lie brackets

We now embark into the computation of Lie brackets.

Proposition 3.35 Let $\mathcal{O} \subset T^*M\otimes T\hat{M}$ be an immersed submanifold, $\overline{T} = (T, \hat{T}), \overline{S} = (S, \hat{S}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$ with $\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q, \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q \in T|_q\mathcal{O}$ for all $q = (x, \hat{x}; A) \in \mathcal{O}$. Then, for every $q \in \mathcal{O}$, one has

$$
\begin{aligned}
[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)), \mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot))]|_q =&\mathscr{L}_{\mathrm{NS}}\big(\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\overline{S} - \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q\overline{T}\big)\big|_q \\
&+ \nu\big(AR(T(q), S(q)) - \hat{R}(\hat{T}(q), \hat{S}(q))A\big)\big|_q, \qquad (25)
\end{aligned}
$$

with both sides tangent to $\mathcal{O}$.

Proof. We will deal first with the case where $\mathcal{O}$ is an open subset of $T^*M \otimes T\hat{M}$. Take a local $\pi_{T^*M\otimes T\hat{M}}]$ section $\tilde{A}$ around $(x, \hat{x})$ such that $\tilde{A}|_{(x,\hat{x})} = A$, $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$; see Lemma 3.24.

Let $f \in C^\infty(T^*M \otimes T\hat{M})$. By using the definition of $\mathscr{L}_{\mathrm{NS}}$ and ν, one obtains

$$\begin{aligned}
&\mathscr{L}_{\mathrm{NS}}(\overline{T}(A))|_q(\mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot)(f)))\\
&= \overline{T}(A)(\mathscr{L}_{\mathrm{NS}}(\overline{S}(\tilde{A}))|_{\tilde{A}}(f)) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \mathscr{L}_{\mathrm{NS}}(\overline{S}(A + t\overline{\nabla}_{\overline{T}(A)}\tilde{A}))|_{A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}}(f)\\
&= \overline{T}(A)\big(\overline{S}(\tilde{A})(f(\tilde{A})) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(\tilde{A} + t\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A})\big)\\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{S}(A + t\overline{\nabla}_{\overline{T}(A)}\tilde{A})(f(\tilde{A} + t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))\\
&\quad + \frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A + t\overline{\nabla}_{\overline{T}(A)}\tilde{A} + s\overline{\nabla}_{\overline{S}(A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A})}(\tilde{A} + t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big)\big).
\end{aligned}$$

We use the fact that $\overline{\nabla}_{\overline{X}}\tilde{A} = 0$ for all $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ and $\frac{\partial}{\partial t}$ and $\overline{T}(\tilde{A})$ commute (as the obvious vector fields on $M \times \hat{M} \times \mathbb{R}$ with points $(x, \hat{x}, t)$) to write the last expression in the form

$$\begin{aligned}
&\overline{T}(A)(\overline{S}(\tilde{A})(f(\tilde{A}))) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{T}(A)(f(\tilde{A} + t\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A})) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{S}(A)(f(\tilde{A} + t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))\\
&+ \frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A + st\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big)\big).
\end{aligned}$$

By interchanging the roles of $\overline{T}$ and $\overline{S}$ and using the definition of commutator of vector fields, we get from this

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)), \mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot))]|_q(f)\\
=&[\overline{T}(\tilde{A}), \overline{S}(\tilde{A})]|_q(f(\tilde{A})) + \frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A + st\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big)\big)\\
&- \frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A + st\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A})\big)\big)\\
=&[\overline{T}(\tilde{A}), \overline{S}(\tilde{A})]|_q(f(\tilde{A})) + \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \nu(t\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))|_q(f)\\
&- \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \nu(t\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A}))|_q(f)\\
=&[\overline{T}(\tilde{A}), \overline{S}(\tilde{A})]|_q(f(\tilde{A})) + \nu(\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))|_q(f) - \nu(\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A}))|_q(f)\\
=&[\overline{T}(\tilde{A}), \overline{S}(\tilde{A})]|_q(f(\tilde{A})) - \nu([\overline{\nabla}_{\overline{T}(\tilde{A})}, \overline{\nabla}_{\overline{S}(\tilde{A})}]\tilde{A}))|_q(f).
\end{aligned}$$

Using Lemma 3.25, we get that the last line is equal to

$$\begin{aligned}
&[\overline{T}(\tilde{A}), \overline{S}(\tilde{A})]|_{(x,\hat{x})}(f(\tilde{A}))\\
&- \nu(\overline{\nabla}_{[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_{(x,\hat{x})}}\tilde{A} - AR(T(A), S(A)) + \hat{R}(\hat{T}(A), \hat{S}(A))A)|_q(f),
\end{aligned}$$

from which, by using the definition of $\mathscr{L}_{\mathrm{NS}}$, linearity of $\nu(\cdot)|_q$ and arbitrariness of $f \in C^\infty(T^*M \otimes T\hat{M})$, we get

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)), \mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot))]|_q = \mathscr{L}_{\mathrm{NS}}([\overline{T}(\tilde{A}), \overline{S}(\tilde{A})])|_q\\
&+ \nu(AR(T(A), S(A)) - \hat{R}(\hat{T}(A), \hat{S}(A))A)|_q.
\end{aligned}$$

Finally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{S}(A + t\underbrace{\overline{\nabla}_{\overline{T}(q)}\tilde{A}}_{=0}) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{S}(A) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{T}(A + t\underbrace{\overline{\nabla}_{\overline{S}(q)}\tilde{A}}_{=0}) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \overline{T}(A) = 0,$$

since $\overline{T}(q), \overline{S}(q) \in T|_{(x,\hat{x})}(M \times \hat{M})$ and hence by Eq. (22),

$$[\overline{T}(\tilde{A}), \overline{S}(\tilde{A})] = \overline{\nabla}_{\overline{T}(q)}(\overline{S}(\tilde{A})) - \overline{\nabla}_{\overline{S}(q)}(\overline{T}(\tilde{A})) = \mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q \overline{S} - \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q \overline{T}.$$

The claim thus holds in this case (i.e. when $\mathcal{O}$ is an open subset of $T^*M \otimes T\hat{M}$). We let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ to be an immersed submanifold and $\overline{T}, \overline{S} : \mathcal{O} \to T(M \times \hat{M})$ are such that, for all $q = (x, \hat{x}; A) \in \mathcal{O}$, $\overline{T}(x, \hat{x}; A)$, $\overline{S}(x, \hat{x}; A)$ belong to $T|_{(x,\hat{x})}(M \times \hat{M})$ and $\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q$, $\mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q$ belong to $T|_q\mathcal{O}$. For a fixed $q = (x, \hat{x}; A) \in \mathcal{O}$, we may, thanks to Lemma 3.27 (taking $\tau = \pi_{T^*M\otimes T\hat{M}}$, $\eta = \pi_{T(M\times\hat{M})}$, $b_0 = q$ and $F = \overline{T}$ or $F = \overline{S}$ there) take a small open neighbourhood V of q in $\mathcal{O}$, a neighbourhood $\tilde{V}$ of q in Q such that $V \subset \tilde{V}$ and some extensions $\tilde{\overline{T}}, \tilde{\overline{S}} : \tilde{V} \to T(M \times \hat{M})$ of $\overline{T}|_V, \overline{S}|_V$ with $\tilde{\overline{T}}(x', \hat{x}'; A'), \tilde{\overline{S}}(x', \hat{x}'; A') \in T|_{(x',\hat{x}')}(M \times \hat{M})$ for all $(x', \hat{x}'; A') \in \tilde{V}$. Then since $\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot))|_V = \mathscr{L}_{\mathrm{NS}}(\tilde{\overline{T}}(\cdot))|_V$, $\mathscr{L}_{\mathrm{NS}}(\overline{S}(\cdot))|_V = \mathscr{L}_{\mathrm{NS}}(\tilde{\overline{S}}(\cdot))|_V$, we compute, because of what has been shown already,

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{NS}}(\overline{T}), \mathscr{L}_{\mathrm{NS}}(\overline{S})]|_q = [\mathscr{L}_{\mathrm{NS}}(\tilde{\overline{T}})|_V, \mathscr{L}_{\mathrm{NS}}(\tilde{\overline{S}})|_V]|_q = ([\mathscr{L}_{\mathrm{NS}}(\tilde{\overline{T}}), \mathscr{L}_{\mathrm{NS}}(\tilde{\overline{S}})]|_V)|_q \\
=&\mathscr{L}_{\mathrm{NS}}\big(\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\tilde{\overline{S}} - \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q\tilde{\overline{T}}\big)\big|_q + \nu\big(AR(T(q), S(q)) - \hat{R}(\hat{T}(q), \hat{S}(q))A\big)\big|_q,
\end{aligned}$$

where in the last line we used that $\tilde{\overline{T}}(q) = \overline{T}(q) = (T(q), \hat{T}(q))$, $\tilde{\overline{S}}(q) = \overline{S}(q) = (S(q), \hat{S}(q))$. Take any $\overline{\omega} \in \Gamma(\pi_{T^m_k(M\times\hat{M})})$. Since $\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q \in T|_q\mathcal{O} = T|_qV$ by assumption and since $(\overline{S}\overline{\omega})|_V = (\tilde{\overline{S}}\overline{\omega})|_V$, we have $\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q(\overline{S}\overline{\omega}) = \mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q(\tilde{\overline{S}}\overline{\omega})|_V$. But then Eq. (23) i.e. the definition of $\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\overline{S}$ implies that

$$\begin{aligned}
&(\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\overline{S})\overline{\omega} = \mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q(\overline{S}\overline{\omega}) - \overline{S}(q)\overline{\nabla}_{\overline{T}(q)}\overline{\omega} \\
=&\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q(\tilde{\overline{S}}\overline{\omega}) - \tilde{\overline{S}}(q)\overline{\nabla}_{\overline{T}(q)}\overline{\omega} = (\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\tilde{\overline{S}})\overline{\omega}
\end{aligned}$$

i.e. $\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\overline{S} = \mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\tilde{\overline{S}}$ and similarly $\mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q\overline{T} = \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q\tilde{\overline{T}}$. This shows that on $\mathcal{O}$ we have the formula

$$\begin{aligned}
[\mathscr{L}_{\mathrm{NS}}(\overline{T}), \mathscr{L}_{\mathrm{NS}}(\overline{S})]|_q =&\mathscr{L}_{\mathrm{NS}}\big(\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q\overline{S} - \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q\overline{T}\big)\big|_q \\
&+ \nu\big(AR(T(q), S(q)) - \hat{R}(\hat{T}(q), \hat{S}(q))A\big)\big|_q,
\end{aligned}$$

where both sides belong to $T|_q\mathcal{O}$ (since the left hand side obviously belongs to $T|_q\mathcal{O}$). $\square$

Proposition 3.36 Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $\overline{T} = (T, \hat{T}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$, $U \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes T\hat{M}})$ be such that, for all $q = (x, \hat{x}; A) \in \mathcal{O}$,

$$\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q \in T|_q\mathcal{O}, \quad \nu(U(q))|_q \in T|_q\mathcal{O}.$$

Then

$$[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)),\nu(U(\cdot))]|_q=-\mathscr{L}_{\mathrm{NS}}(\nu(U(q))|_q\overline{T})|_q+\nu(\mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_qU)|_q,$$

with both sides tangent to $\mathcal{O}$.

Proof. As in the proof of Proposition 3.35, we will deal first with the case where $\mathcal{O}$ is an open subset of $T^*M\otimes T\hat{M}$. Take a local $\pi_{T^*M\otimes T\hat{M}}]$ section $\tilde{A}$ around $(x,\hat{x})$ such that $\tilde{A}|_{(x,\hat{x})}=A$, $\overline{\nabla}\tilde{A}|_{(x,\hat{x})}=0$; see Lemma 3.24. In some expressions we will write $q=A$ for clarity.

Let $f\in C^\infty(T^*M\otimes T\hat{M})$. Then $\mathscr{L}_{\mathrm{NS}}(\overline{T}(A))|_q\big(\nu(U(\cdot))(f)\big)$ is equal to

$$\overline{T}(A)\big(\nu(U(\tilde{A}))\big|_{\tilde{A}}(f)\big)-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\nu(U(A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}))\big|_{A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}}(f),$$

which is equal to $\overline{T}(A)\big(\nu(U(\tilde{A}))\big|_{\tilde{A}}(f)\big)$ once we recall that $\overline{\nabla}_{\overline{T}(A)}\tilde{A}=0$. In addition, one has

$$\begin{aligned}
&\nu(U(A))|_q\big(\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot))(f)\big)=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\mathscr{L}_{\mathrm{NS}}(\overline{T}(A+tU(A))\big|_{A+tU(A)}(f)\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A+tU(A))\big(f(\tilde{A}+tU(\tilde{A}))\big)\\
&-\frac{\partial^2}{\partial s\partial t}\big|_0f\big(A+tU(A)+s\overline{\nabla}_{\overline{T}(A+tU(A))}(\tilde{A}+tU(\tilde{A}))\big)\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A+tU(A))\big(f(\tilde{A}+tU(\tilde{A}))\big)-\frac{\partial^2}{\partial s\partial t}\big|_0f\big(A+tU(A)+st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big),
\end{aligned}$$

since $\overline{\nabla}_{\overline{T}(A+tU(A))}\tilde{A}=0$. We next simplify the first term on the last line to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A+tU(A))\big(f(\tilde{A}+tU(\tilde{A}))\big)\\
=&(\nu(U(q))|_q\overline{T})\big(f(\tilde{A})\big)+\overline{T}(A)\big(\nu(U(\tilde{A}))|_{\tilde{A}}(f)\big)
\end{aligned}$$

and then, for the second term, one obtains

$$\begin{aligned}
&\frac{\partial^2}{\partial s\partial t}\big|_0f\big(A+tU(A)+st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big)\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0f_*|_q\nu\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big(tU(A)+st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big)\Big)\Big|_q\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0f_*|_q\nu\big(U(A)+s\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0\Big(f_*|_q\nu(U(A))|_q+sf_*|_q\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)|_q\Big)\\
=&f_*\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)|_q=\nu(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A})))|_qf.
\end{aligned}$$

Therefore one deduces

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{NS}}(\overline{T}(\cdot)),\nu(U(\cdot))]|_q(f)=-(\nu(U(q))|_q\overline{T})\big(f(\tilde{A})\big)+\nu(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A})))|_qf\\
=&-\tilde{A}_*(\nu(U(A))|_q\overline{T})(f)+\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q(f)\\
=&-\mathscr{L}_{\mathrm{NS}}(\nu(U(A))|_q\overline{T})|_q(f)+\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q(f),
\end{aligned}$$

where the last line follows from the definition of $\mathscr{L}_{\rm NS}$ and the fact that $\overline{\nabla}_{\nu(U(A))|_q\overline{T}}\tilde{A}=0$. Finally, Eq. (22) implies

$$\overline{\nabla}_{T(q)}(U(\tilde{A}))=\overline{\nabla}_{T(q)}(U(\tilde{A}))-\nu(\underbrace{\overline{\nabla}_{T(q)}\tilde{A}}_{=0})|_qU=\mathscr{L}_{\rm NS}(\overline{T}(A))|_qU.$$

Thus the claimed formula holds in the special case where $\mathcal{O}$ is an open subset of $T^*M\otimes T\hat{M}$. More generally, let $\mathcal{O}\subset T^*M\otimes T\hat{M}$ be an immersed submanifold, and $\overline{T}=(T,\hat{T}):\mathcal{O}\to T(M\times\hat{M})=TM\times T\hat{M}$, $U:\mathcal{O}\to T^*M\times T\hat{M}$ as in the statement of this proposition.

For a fixed $q=(x,\hat{x};A)\in\mathcal{O}$, Lemma 3.27 implies the existence of a neighbourhood V of q in $\mathcal{O}$, a neighbourhood $\tilde{V}$ of q in $T^*M\otimes T\hat{M}$ and smooth $\tilde{\overline{T}}:\tilde{V}\to T(M\times\hat{M})$, $\tilde{U}:\tilde{V}\to T^*M\otimes T\hat{M}$ such that $\tilde{\overline{T}}(x,\hat{x};A)\in T|_{(x,\hat{x})}(M\times\hat{M})$, $\tilde{U}(x,\hat{x};A)\in(T^*M\otimes T\hat{M})|_{(x,\hat{x})}$ and $\tilde{\overline{T}}|_V=\overline{T}|_V$, $\tilde{U}|_V=U|_V$ (for the case of an extension $\tilde{U}$ of U, take in Lemma 3.27, $\tau=\pi_{T^*M\otimes T\hat{M}}$, $\eta=\pi_{T^1_1(M\times\hat{M})}$, $F=U$, $b_0=q$). In the same way as in the proof of Proposition 3.35, we have $[\mathscr{L}_{\rm NS}(\overline{T}),\nu(U)]|_q=[\mathscr{L}_{\rm NS}(\tilde{\overline{T}}),\nu(\tilde{U})]_q$ and $\mathscr{L}_{\rm NS}(\tilde{\overline{T}}(q))|_q\tilde{U}=\mathscr{L}_{\rm NS}(\overline{T}(q))|_qU$. Hence by what was already shown above,

$$[\mathscr{L}_{\rm NS}(\overline{T}),\nu(U)]|_q=-\mathscr{L}_{\rm NS}(\nu(U(q))|_q\tilde{\overline{T}})|_q+\nu(\mathscr{L}_{\rm NS}(\overline{T}(q))|_qU)|_q.$$

We are left to show that $\nu(U(q))|_q\tilde{\overline{T}}=\nu(U(q))|_q\overline{T}$ and for that, it suffices to show that $\nu(\nu(U(q))|_q\tilde{\overline{T}})|_{\overline{T}(q)}=\nu(\nu(U(q))|_q\overline{T})|_{\overline{T}(q)}$: if $f\in C^\infty(T(M\times\hat{M}))$, then

$$\begin{aligned}\nu(\nu(U(q))|_q\tilde{\overline{T}})|_{\overline{T}(q)}f=&\big(\tilde{\overline{T}}_*\nu(U(q))|_q\big)f=\nu(U(q))|_q(f\circ\tilde{\overline{T}})=\nu(U(q))|_q(f\circ\overline{T})\\=&\big(\overline{T}_*\nu(U(q))|_q\big)f=\nu(\nu(U(q))|_q\overline{T})|_{\overline{T}(q)}f,\end{aligned}$$

where at the 3rd equality we used the fact that $(f\circ\tilde{\overline{T}})|_V=(f\circ\overline{T})|_V$ and $\nu(U(q))|_q\in T|_q\mathcal{O}=T|_qV$. This completes the proof.

□

Finally, we derive a formula for the commutators of two vertical vector fields.

Proposition 3.37 Let $\mathcal{O}\subset T^*M\otimes T\hat{M}$ be an immersed submanifold and $U,V\in C^\infty(\pi_{\mathcal{O}},\pi_{T^*M\otimes T\hat{M}})$ be such that $\nu(U(q))|_q,\nu(V(q))|_q\in T|_q\mathcal{O}$ for all $q\in\mathcal{O}$. Then

$$[\nu(U(\cdot)),\nu(V(\cdot))]|_q=\nu\big(\nu(U(q))|_qV-\nu(V(q))|_qU\big)|_q. \tag{26}$$

Proof. Again we begin with the case where $\mathcal{O}$ is an open subset of $T^*M\otimes T\hat{M}$ and write $q=(x,\hat{x};A)\in\mathcal{O}$ simply as A. Let $f\in C^\infty(T^*M\otimes T\hat{M})$. Then,

$$\begin{aligned}&\nu(U(A))|_q\big(\nu(V(\cdot))(f)\big)=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\nu(V(A+tU(A))|_{A+tU(A)}(f)\\=&\frac{\partial^2}{\partial t\partial s}\big|_0f(A+tU(A)+sV(A+tU(A)))\\=&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0f_*|_q\nu\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big(tU(A)+sV(A+tU(A))\big)\Big)\Big|_q\\=&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0f_*\nu\big(U(A)+s\nu(U(A))|_qV\big)|_q\\=&f_*\nu\big(\nu(U(A))|_qV\big)|_q=\nu(\nu(U(A))|_qV)|_qf.\end{aligned}$$

from which the result follows in the case that $\mathcal{O}$ is an open subset of $T^*M \otimes T\hat{M}$. The case where $\mathcal{O}$ is only an immersed submanifold of $T^*M \otimes T\hat{M}$ can be treated by using Lemma 3.27 in the same way as in the proofs of Propositions 3.35, 3.36.

$\square$

As a corollary to the previous three propositions, we have the following, whose proof is immediate.

Corollary 3.38 Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold and $\mathcal{X}, \mathcal{Y} \in \mathrm{VF}(\mathcal{O})$. Letting for $q \in \mathcal{O}$,

$$\mathcal{X}|_q = \mathscr{L}_{\mathrm{NS}}(\overline{T}(q))|_q + \nu(U(q))|_q, \quad \mathcal{Y}|_q = \mathscr{L}_{\mathrm{NS}}(\overline{S}(q))|_q + \nu(V(q))|_q,$$

to be the unique decompositions given by Proposition 3.30. Writing $\overline{T} = (T, \hat{T})$, $\overline{S} = (S, \hat{S})$ corresponding to $T(M \times \hat{M}) = TM \times T\hat{M}$, we get

$$\begin{aligned}
[\mathcal{X}, \mathcal{Y}]|_q =& \big(\mathscr{L}_{\mathrm{NS}}(\mathcal{X}|_q\overline{S})|_q + \nu(\mathcal{X}|_q V)|_q\big) - \big(\mathscr{L}_{\mathrm{NS}}(\mathcal{Y}|_q\overline{T})|_q + \nu(\mathcal{Y}|_q U)|_q\big) \\
&+ \nu\big(AR(T(q), S(q)) - \hat{R}(\hat{T}(q), \hat{S}(q))A\big)|_q
\end{aligned}$$

(for the notation, see the second remark after Proposition 3.30).

4 Study of the Rolling problem (R)

In this section, we investigate the rolling problem as a control system $(\Sigma)_R$ associated to the distribution $\mathcal{D}_{\mathrm{R}}$.

4.1 Global properties of a $\mathcal{D}_{\mathrm{R}}$-orbit

We begin with the following remark.

Remark 4.1 Notice that the map $\pi_{Q,M} : Q \to M$ is in fact a bundle. Indeed, let $F = (X_i)_{i=1}^n$ be a local oriented orthonormal frame of M defined on an open set U. Then the local trivialization of $\pi_{Q,M}$ induced by F is

$$\tau_F : \pi_{Q,M}^{-1}(U) \to U \times F_{\mathrm{OON}}(\hat{M}); \quad \tau_F(x, \hat{x}; A) = (x, (AX_i|_x)_{i=1}^n),$$

is a diffeomorphism. Note also that since $\pi_{Q,M}$-fibers are diffeomorphic to $F_{\mathrm{OON}}(\hat{M})$, in order that there would be a principal G-bundle structure for $\pi_{Q,M}$, it is necessary that $F_{\mathrm{OON}}(\hat{M})$ is diffeomorphic to the Lie-group G.

From Proposition 3.20, we deduce that each $\mathcal{D}_{\mathrm{R}}$-orbit is a smooth bundle over M. This is given in the next proposition.

Proposition 4.2 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and suppose that $\hat{M}$ is complete. Then

$$\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0), M} := \pi_{Q,M}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to M,$$

is a smooth subbundle of $\pi_{Q,M}$.

One defines similarly $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0), \hat{M}} := \pi_{Q,\hat{M}}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to \hat{M}$.

Proof. First, surjectivity of $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ follows from completeness of $\hat{M}$ by using Proposition 3.18. Since $\mathcal{D}_{\mathrm{R}}|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for every $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and $(\pi_{Q,M})_*$ maps $\mathcal{D}_{\mathrm{R}}|_q$ isomorphically onto $T|_{\pi_{Q,M}(q)}M$, one immediately deduces that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is also a submersion. This implies that each fiber $(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(x)$, $x \in M$, is a smooth closed submanifold of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Choose next, for each $x \in M$, an open convex $U_x \subset T|_xM$ such that $\exp_x|_{U_x}$ is a diffeomorphism onto its image and $0 \in U$. Define

$$\begin{aligned}\tau_x : \pi_{Q,M}^{-1}(U_x) &\to U_x \times \pi_{Q,M}^{-1}(x),\\ q = (y, \hat{y}; A) &\mapsto \big(y, \big(x, \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma_{y,x}, q)(1); A_{\mathcal{D}_{\mathrm{R}}}(\gamma_{y,x}, q)(1)\big)\big),\end{aligned}$$

where $\gamma_{y,x} : [0,1] \to M$; $\gamma_{y,x}(t) = \exp_x((1-t)\exp_x^{-1}(y))$ is a geodesic from y to x. It is obvious that τ_x is a smooth bijection. Moreover, restricting τ_x to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ clearly gives a smooth bijection

$$\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(U_x) \to U_x \times (\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(x)).$$

The inverse of τ_x, $\tau_x^{-1} : U_x \times \pi_{Q,M}^{-1}(x) \to \pi_{Q,M}^{-1}(U_x)$ is constructed with a formula similar to that of τ_x and is seen, in the same way, to be smooth. This inverse restricted to $U_x \times (\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(x))$ maps bijectively onto $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_{Q,M}^{-1}(U_x)$ and thus τ_x is a smooth local trivialization of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. This completes the proof. □

Remark 4.3 In the case where $\hat{M}$ is not complete, the result of Proposition 4.2 remains valid if we just claim that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is a bundle over its image $M^\circ := \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, which is an open connected subset of M.

Write $\hat{M}^\circ := \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Then using the diffeomorphism $\iota : Q := Q(M,\hat{M}) \to \hat{Q} := Q(\hat{M}, M)$; $(x, \hat{x}; A) \mapsto (\hat{x}, x; A^{-1})$ (Proposition 3.22) one gets

$$\begin{aligned}\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}} =& \pi_{Q,\hat{M}}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)} = \pi_{Q,\hat{M}} \circ \iota^{-1}|_{\mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\iota(q_0))} \circ \iota|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}\\ =& \pi_{\hat{Q},\hat{M}}|_{\mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\iota(q_0))} \circ \iota|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)} = \pi_{\mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\iota(q_0)),\hat{M}} \circ \iota|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)},\end{aligned}$$

from which we see that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}}$ is also a bundle over its image $\hat{M}^\circ$ since $\iota|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to \mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\iota(q_0))$ is a diffeomorphism and since by the previous proposition and the above remark $\pi_{\mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\iota(q_0)),\hat{M}}$ is a bundle over its image, which necessarily is $\hat{M}^\circ$. Notice also that if M is complete, then $\hat{M}^\circ = \hat{M}$.

The next proposition can be useful in case one of the manifolds has a large group of isometries. We do not provide an argument for this proposition since it is immediate.

Proposition 4.4 For any Riemannian isometries $F \in \mathrm{Iso}(M,g)$ and $\hat{F} \in \mathrm{Iso}(\hat{M},\hat{g})$ of (M,g), $(\hat{M},\hat{g})$ respectively, one defines smooth free right and left actions of $\mathrm{Iso}(M,g)$, $\mathrm{Iso}(\hat{M},\hat{g})$ on Q by

$$q_0 \cdot F := (F^{-1}(x_0), \hat{x}_0; A_0 \circ F_*|_{F^{-1}(x_0)}), \quad \hat{F} \cdot q_0 := (x_0, \hat{F}(\hat{x}_0); \hat{F}_*|_{\hat{x}_0} \circ A_0),$$

where $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. We also set

$$\hat{F} \cdot q_0 \cdot F := (\hat{F} \cdot q_0) \cdot F = \hat{F} \cdot (q_0 \cdot F).$$

Then for any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, a.c. $\gamma : [0,1] \to M$, $\gamma(0) = x_0$, and $F \in \mathrm{Iso}(M,g)$, $\hat{F} \in \mathrm{Iso}(\hat{M},\hat{g})$, one has

$$\hat{F} \cdot q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) \cdot F = q_{\mathcal{D}_{\mathrm{R}}}(F^{-1} \circ \gamma, \hat{F} \cdot q_0 \cdot F)(t), \tag{27}$$

for all $t \in [0,1]$. In particular,

$$\hat{F} \cdot \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cdot F = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\hat{F} \cdot q_0 \cdot F).$$

We derive the following consequence.

Corollary 4.5 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and $\gamma, \omega : [0,1] \to M$ be absolutely continuous such that $\gamma(0) = \omega(0) = x_0$, $\gamma(1) = \omega(1)$. Then assuming that $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)$, $q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_0)$, $q_{\mathcal{D}_{\mathrm{R}}}(\omega^{-1}.\gamma, q_0)$ exist and if there exists $\hat{F} \in \mathrm{Iso}(\hat{M},\hat{g})$ such that

$$\hat{F} \cdot q_0 = q_{\mathcal{D}_{\mathrm{R}}}(\omega^{-1}.\gamma, q_0)(1),$$

then

$$\hat{F} \cdot q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_0)(1) = q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1).$$

Proof.

$$\begin{aligned} & q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1) = q_{\mathcal{D}_{\mathrm{R}}}(\omega.\omega^{-1}.\gamma, q_0)(1) = \big(q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_{\mathcal{D}_{\mathrm{R}}}(\omega^{-1}.\gamma, q_0)(1)).q_{\mathcal{D}_{\mathrm{R}}}(\omega^{-1}.\gamma, q_0)\big)(1) \\ =& \big(q_{\mathcal{D}_{\mathrm{R}}}(\omega, \hat{F} \cdot q_0).q_{\mathcal{D}_{\mathrm{R}}}(\omega^{-1}.\gamma, q_0)\big)(1) = q_{\mathcal{D}_{\mathrm{R}}}(\omega, \hat{F} \cdot q_0)(1) = \hat{F} \cdot q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_0)(1). \end{aligned}$$

□

Proposition 4.6 Let $\pi : (M_1, g_1) \to (M,g)$ and $\hat{\pi} : (\hat{M}_1, \hat{g}_1) \to (\hat{M}, \hat{g})$ be Riemannian coverings. Write $Q_1 = Q(M_1, \hat{M}_1)$ and $(\mathcal{D}_{\mathrm{R}})_1$ for the rolling distribution in Q_1. Then the map

$$\Pi : Q_1 \to Q; \quad \Pi(x_1, \hat{x}_1; A_1) = \big(\pi(x_1), \hat{\pi}(\hat{x}_1); \hat{\pi}_*|_{\hat{x}_1} \circ A_1 \circ (\pi_*|_{x_1})^{-1}\big)$$

is a covering map of Q_1 over Q and

$$\Pi_*(\mathcal{D}_{\mathrm{R}})_1 = \mathcal{D}_{\mathrm{R}}.$$

Moreover, for every $q_1 \in Q_1$ the restriction onto $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)$ of Π is a covering map $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) \to \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$. Then, for every $q_1 \in Q_1$, $\Pi(\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$ and one has $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) = Q_1$ if and only if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1)) = Q$.

As an immediate corollary of the above proposition, we obtain the following result regarding the complete controllability of $(\mathcal{D}_{\mathrm{R}})$.

Corollary 4.7 Let $\pi : (M_1, g_1) \to (M,g)$ and $\hat{\pi} : (\hat{M}_1, \hat{g}_1) \to (\hat{M}, \hat{g})$ be Riemannian coverings. Write $Q = Q(M, \hat{M})$, $\mathcal{D}_{\mathrm{R}}$ and $Q_1 = Q(M_1, \hat{M}_1)$, $(\mathcal{D}_{\mathrm{R}})_1$ respectively for the state space and for the rolling distribution in the respective state space. Then the control system associated to $\mathcal{D}_{\mathrm{R}}$ is completely controllable if and only if the control system associated to $(\mathcal{D}_{\mathrm{R}})_1$ is completely controllable. As a consequence, when one addresses the complete controllability issue for the rolling distribution $\mathcal{D}_{\mathrm{R}}$, one can assume with no loss of generality that both manifolds M and $\hat{M}$ are simply connected.

We now proceed with the proof of Proposition 4.6.

Proof. It is clear that Π is a local diffeomorphism onto Q. To show that it is a covering map, let $q_1=(x_1,\hat{x}_1;A_1)$ and choose evenly covered w.r.t π, $\hat{\pi}$ open sets U and $\hat{U}$ of M, $\hat{M}$ containing $\pi(x_1)$, $\hat{\pi}(\hat{x}_1)$, respectively. Thus $\pi^{-1}(U)=\bigcup_{i\in I}U_i$ and $\hat{\pi}^{-1}(\hat{U})=\bigcup_{i\in\hat{I}}\hat{U}_i$ where U_i, $i\in I$ (resp. $\hat{U}_i$, $i\in\hat{I}$) are mutually disjoint connected open subsets of M_1 (resp. $\hat{M}_1$) such that π (resp. $\hat{\pi}$) maps each U_i (resp. $\hat{U}_i$) diffeomorphically onto U (resp. $\hat{U}$). Then

$$\Pi^{-1}(\pi_Q^{-1}(U\times\hat{U}))=\pi_{Q_1}^{-1}((\pi\times\hat{\pi})^{-1}(U\times\hat{U}))=\bigcup_{i\in I,j\in\hat{I}}\pi_{Q_1}^{-1}(U_i\times\hat{U}_j),$$

where $\pi_{Q_1}^{-1}(U_i\times\hat{U}_j)$ for $(i,j)\in I\times\hat{I}$ are clearly mutually disjoint and connected. Now if for a given $(i,j)\in I\times\hat{I}$ we have $(y_1,\hat{y}_1,B_1),(z_1,\hat{z}_1;C_1)\in\pi_{Q_1}^{-1}(U_i\times\hat{U}_j)$ such that $\Pi(y_1,\hat{y}_1;B_1)=\Pi(z_1,\hat{z}_1,C_1)$, then $y_1=z_1$, $\hat{y}_1=\hat{z}_1$ and hence $B_1=C_1$, which shows that Π restricted to $\pi_{Q_1}^{-1}(U_i\times\hat{U}_j)$ is injective. It is also a local diffeomorphism, as mentioned above, and clearly surjective onto $\pi_Q^{-1}(U\times\hat{U})$, which proves that $\pi_Q^{-1}(U\times\hat{U})$ is evenly covered with respect to Π. This finishes the proof that Π is a covering map. Suppose next that $q_1(t)=(\gamma_1(t),\hat{\gamma}_1(t);A_1(t))$ is a smooth path on Q_1 tangent to $(\mathcal{D}_{\mathrm{R}})_1$ and defined on an interval containing $0\in\mathbb{R}$. Define $q(t)=(\gamma(t),\hat{\gamma}(t);A(t)):=(\Pi\circ q_1)(t)$. Then

$$\begin{aligned}
\dot{\hat{\gamma}}(t)=&\hat{\pi}_*\dot{\hat{\gamma}}_1(t)=\hat{\pi}_*A_1(t)\dot{\gamma}_1(t)=A(t)\pi_*\dot{\gamma}_1(t)=A(t)\dot{\gamma}(t)\\
A(t)=&\hat{\pi}_*|_{\hat{\gamma}_1(t)}\circ P_0^t(\hat{\gamma}_1(t))\circ A_1(0)\circ P_t^0(\gamma_1)\circ(\pi_*|_{\gamma_1(t)})^{-1}\\
=&P_0^t(\hat{\gamma}(t))\circ\hat{\pi}_*|_{\hat{\gamma}_1(t)}\circ A_1(0)\circ(\pi_*|_{\gamma_1(t)})^{-1}\circ P_t^0(\gamma)\\
=&P_0^t(\hat{\gamma}(t))\circ A(0)\circ P_t^0(\gamma),
\end{aligned}$$

which shows that $q(t)$ is tangent to $\mathcal{D}_{\mathrm{R}}$. This shows that $\Pi_*(\mathcal{D}_{\mathrm{R}})_1\subset\mathcal{D}_{\mathrm{R}}$ and the equality follows from the fact that Π is a local diffeomorphism and the ranks of $(\mathcal{D}_{\mathrm{R}})_1$ and $\mathcal{D}_{\mathrm{R}}$ are the same i.e., $=n$.

Let $q_1=(x_1,\hat{x}_1;A_1)$. We proceed to show that the restriction of Π gives a covering $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)\to\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$. First, since $\Pi_*(\mathcal{D}_{\mathrm{R}})_1=\mathcal{D}_{\mathrm{R}}$ and $\Pi:Q_1\to Q$ is a covering map, it follows that $\Pi(\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1))=\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$. Let $q:=\Pi(q_1)$ and let $U\subset Q$ be an evenly covered neighbourhood of q w.r.t. Π. By the Orbit Theorem, there exists vector fields $Y_1,\dots,Y_d\in\mathrm{VF}(Q)$ tangent to $\mathcal{D}_{\mathrm{R}}$ and $(u_1,\dots,u_d)\in(L^1([0,1]))^d$ and a connected open neighbourhood W of $(u_1,\dots,u_d)$ in $(L^1([0,1]))^d$ such that the image of the end point map $\mathrm{end}_{(Y_1,\dots,Y_d)}(q,W)$ is an open subset of the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ containing q and included in the Π-evenly covered set U. Let $(Y_i)_1$, $i=1,\dots,d$, be the unique vector fields on Q_1 defined by $\Pi_*(Y_i)_1=Y_i$, $i=1,\dots,d$. Since $\Pi_*(\mathcal{D}_{\mathrm{R}})_1=\mathcal{D}_{\mathrm{R}}$, it follows that $(Y_i)_1$ are tangent to $(\mathcal{D}_{\mathrm{R}})_1$ and also, $\Pi\circ\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}=\mathrm{end}_{(Y_1,\dots,Y_d)}\circ(\Pi\times\mathrm{id})$. It follows that $\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}(q_1',W)$ is an open subset of $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)$ contained in $\Pi^{-1}(U)$ for every $q_1'\in(\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)})^{-1}(q)$. Since $\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}$ is continuous and W is connected, it thus follows that for each $q_1'\in(\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)})^{-1}(q)$, the connected set $\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}(q_1',W)$ is contained in a single component of $\Pi^{-1}(U)$ which, since U was evenly covered, is mapped diffeomorphically by Π onto U. But then Π maps $\mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}(q_1',W)$ diffeomorphically onto $\mathrm{end}_{(Y_1,\dots,Y_d)}(q,W)$. Since it is also obvious that

$$(\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)})^{-1}\big(\mathrm{end}_{(Y_1,\dots,Y_d)}(q,W)\big) = \bigcup_{q_1'\in(\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)})^{-1}(q)} \mathrm{end}_{((Y_1)_1,\dots,(Y_d)_1)}(q_1',W),$$

we have proved that $\mathrm{end}_{(Y_1,\dots,Y_d)}(q,W)$ is an evenly covered neighbourhood of q in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ w.r.t $\Pi|_{\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)}$.

Finally, let us prove that for every $q_1 \in Q_1$, the following implication holds true,

$$\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1)) = Q \implies \mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) = Q_1,$$

(the converse statement being trivial). Indeed, if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1)) = Q$, then, for every $q \in Q$, $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) = Q$ and, on the other hand, the fact that Π restricts to a covering map $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1') \to \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1')) = Q$ for any $q_1' \in Q_1$ implies that all the orbits $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1')$, $q_1' \in Q_1$, are open on Q_1. But Q_1 is connected (and orbits are non-empty) and hence there cannot be but one orbit. In particular, $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1) = Q_1$. □

4.2 Rolling Curvature and Lie Algebraic Structure of $\mathcal{D}_{\mathrm{R}}$

4.2.1 Rolling Curvature

We compute some commutators of the vector fields of the form $\mathscr{L}_{\mathrm{R}}(X)$ with $X \in \mathrm{VF}(M)$. The formulas obtained hold both in Q and $T^*M \otimes T\hat{M}$ and thus we do them in the latter space.

The first commutators of the $\mathcal{D}_{\mathrm{R}}$-lifted fields are given in the following theorem.

Proposition 4.8 If $X, Y \in \mathrm{VF}(M)$, $q = (x_0, \hat{x}_0; A) \in T^*(M) \otimes T(\hat{M})$, then the commutator of the lifts $\mathscr{L}_{\mathrm{R}}(X)$ and $\mathscr{L}_{\mathrm{R}}(Y)$ at q is given by

$$[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]|_q = \mathscr{L}_{\mathrm{R}}([X,Y])|_q + \nu(AR(X,Y) - \hat{R}(AX,AY)A)|_q. \tag{28}$$

Proof. Choosing $\overline{T}(B) = (X, BX)$, $\overline{S}(B) = (Y, BY)$ for $B \in T^*(M) \otimes T(\hat{M})$ in proposition 3.35 we have

$$\begin{aligned}[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]|_q = &\mathscr{L}_{\mathrm{NS}}(\mathscr{L}_{\mathrm{NS}}(X,AX)|_q\overline{S} - \mathscr{L}_{\mathrm{NS}}(Y,AY)|_q\overline{T})|_q \\ &+ \nu(AR(X,Y) - \hat{R}(AX,AY)A)|_q.\end{aligned}$$

By Lemma 3.33 one has

$$\begin{aligned}\mathscr{L}_{\mathrm{NS}}(X,AX)|_q\overline{S} =&\mathscr{L}_{\mathrm{NS}}(X,AX)|_q(Y + (\cdot)Y) = \mathscr{L}_{\mathrm{NS}}(X,AX)|_qY + A\mathscr{L}_{\mathrm{NS}}(X,AX)|_qY \\ =&\nabla_X Y + A\nabla_X Y,\end{aligned}$$

so

$$\begin{aligned}\mathscr{L}_{\mathrm{NS}}(\mathscr{L}_{\mathrm{NS}}(X,AX)|_q\overline{S} - \mathscr{L}_{\mathrm{NS}}(Y,AY)|_q\overline{T})|_q =&\mathscr{L}_{\mathrm{NS}}(\nabla_X Y + A\nabla_X Y - \nabla_Y X - A\nabla_Y X)|_q \\ =&\mathscr{L}_{\mathrm{R}}(\nabla_X Y - \nabla_Y X)|_q,\end{aligned}$$

which proves the claim after noticing that, by torsion freeness of ∇, one has $\nabla_X Y - \nabla_Y X = [X,Y]$. □

Proposition 4.8 justifies the next definition.

Definition 4.9 Given vector fields $X, Y, Z_1, \ldots, Z_k \in \mathrm{VF}(M)$, we define the *Rolling Curvature* of the rolling of M against $\hat{M}$ as the smooth mapping

$$\mathsf{Rol}(X, Y) : \pi_{T^*M \otimes T\hat{M}} \to \pi_{T^*M \otimes T\hat{M}},$$

by

$$\mathsf{Rol}(X, Y)(A) := AR(X, Y) - \hat{R}(AX, AY)A, . \tag{29}$$

Moreover, for $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, we use Rol_q to denote the linear map $\wedge^2 T|_x M \to T^*|_x M \otimes T|_{\hat{x}} \hat{M}$ defined on pure elements of $\wedge^2 T|_x M$ by

$$\mathsf{Rol}_q(X \wedge Y) = \mathsf{Rol}(X, Y)(A). \tag{30}$$

Similarly, for $k \geq 0$, we define the smooth mapping

$$\overline{\nabla}^k \mathsf{Rol}(X, Y, Z_1, \ldots, Z_k) : \pi_{T^*M \otimes T\hat{M}} \to \pi_{T^*M \otimes T\hat{M}},$$

by

$$\begin{aligned} \overline{\nabla}^k \mathsf{Rol}(X, Y, Z_1, \ldots, Z_k)(A) := {} & A\nabla^k R(X, Y, (\cdot), Z_1, \ldots, Z_k) \\ & - \hat{\nabla}^k \hat{R}(AX, AY, A(\cdot), AZ_1, \ldots, AZ_k). \end{aligned} \tag{31}$$

Restricting to Q, we have

$$\mathsf{Rol}(X, Y), \overline{\nabla}^k \mathsf{Rol}(X, Y, Z_1, \ldots, Z_k)(A) \in C^\infty(\pi_Q, \pi_{T^*M \otimes T\hat{M}}),$$

such that, for all $(x, \hat{x}; A) \in Q$,

$$\mathsf{Rol}(X, Y)(A), \overline{\nabla}^k \mathsf{Rol}(X, Y, Z_1, \ldots, Z_k)(A) \in A(\mathfrak{so}(T|_x M)).$$

Remark 4.10 With this notation, Eq. (28) of Proposition 4.8 can be written as

$$[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]|_q = \mathscr{L}_{\mathrm{R}}([X, Y])|_q + \nu(\mathsf{Rol}(X, Y)(A))|_q.$$

Recall that using the metric g, one may identify $T^*|_x M \wedge T|_x M = \mathfrak{so}(T|_x M)$ with $\wedge^2 T|_x M$ as we usually do without mention. In order to take advantage of the spectral properties of a (real) symmetric endomorphism, we introduce the following operator associated to the rolling curvature.

Definition 4.11 If $q = (x, \hat{x}; A) \in Q$, let $\widetilde{\mathsf{Rol}}_q : \wedge^2 T|_x M \to \wedge^2 T|_x M$ be the (real) symmetric endomorphism defined by

$$\widetilde{\mathsf{Rol}}_q := A^{\overline{T}} \mathsf{Rol}_q. \tag{32}$$

In particular, eigenvalues of $R|_x$, $\hat{R}|_{\hat{x}}$ and $\widetilde{\mathsf{Rol}}_q$ are real and the eigenspaces corresponding to distinct eigenvalues are orthogonal one to the other.

4.2.2 Computation of more Lie brackets

Proposition 4.12 Let $X, Y, Z \in \mathrm{VF}(M)$. Then, for $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$, one has

$$[\mathscr{L}_{\mathrm{R}}(Z), \nu(\mathsf{Rol}(X,Y)(\cdot))]|_q = - \mathscr{L}_{\mathrm{NS}}(\mathsf{Rol}(X,Y)(A)Z)|_q + \nu\big(\overline{\nabla}^1 \mathsf{Rol}(X,Y,Z)(A)\big)\big|_q + \nu\big(\mathsf{Rol}(\nabla_Z X, Y)(A)\big)\big|_q + \nu\big(\mathsf{Rol}(X, \nabla_Z Y)(A)\big)\big|_q.$$

Proof. Taking $\overline{T}(B) = (Z, BZ)$ and $U = \mathsf{Rol}(X,Y)$ for $B \in T^*M \otimes T\hat{M}$ in Proposition 3.36, we get

$$\begin{aligned}&[\mathscr{L}_{\mathrm{R}}(Z), \nu(\mathsf{Rol}(X,Y)(\cdot))]|_q \\ =& - \mathscr{L}_{\mathrm{NS}}(\nu(\mathsf{Rol}(X,Y)(A))|_q(Z + (\cdot)Z))|_q + \nu(\mathscr{L}_{\mathrm{NS}}(Z, AZ)|_q \mathsf{Rol}(X,Y)(\cdot))|_q.\end{aligned}$$

By Lemma 3.33 one has

$$\nu(\mathsf{Rol}(X,Y)(A))|_q(Z + (\cdot)Z) = \mathsf{Rol}(X,Y)(A)Z,$$

while by taking a local $\pi_{T^*M \otimes T\hat{M}}$-section $\tilde{A}$ s. t. $\tilde{A}|_{(x,\hat{x})} = A$, $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$, one gets

$$\begin{aligned}\mathscr{L}_{\mathrm{NS}}(Z, AZ)|_q \mathsf{Rol}(X,Y)(\cdot) =& \overline{\nabla}_{Z + AZ}(\mathsf{Rol}(X,Y)(\tilde{A})) \\ =& \overline{\nabla}^1 \mathsf{Rol}(X,Y,Z)(A) + \mathsf{Rol}(\nabla_Z X, Y)(A) + \mathsf{Rol}(X, \nabla_Z Y)(A).\end{aligned}$$

$\square$

By Proposition 4.8, the last two terms (when considered as vector fields on $T^*M \otimes T\hat{M}$) on the right hand side belong to $\mathrm{VF}^2_{\mathcal{D}_{\mathrm{R}}}$.

Since for $X, Y \in \mathrm{VF}(M)$ and $q = (x, \hat{x}; A) \in Q$ we have $\nu(\mathsf{Rol}(X,Y)(A))|_q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ by Proposition 4.8, it is reasonable to compute the Lie-bracket of two elements of this type. This is given in the following proposition.

Proposition 4.13 For any $q = (x, \hat{x}; A) \in Q$ and $X, Y, Z, W \in \mathrm{VF}(M)$ we have

$$\begin{aligned}&\big[\nu(\mathsf{Rol}(X,Y)(\cdot)), \nu(\mathsf{Rol}(Z,W)(\cdot))\big]\big|_q \\ =& \nu\big(\mathsf{Rol}(X,Y)(A)R(Z,W) - \hat{R}(\mathsf{Rol}(X,Y)(A)Z, AW)A - \hat{R}(AZ, \mathsf{Rol}(X,Y)(A)W)A \\ &- \hat{R}(AZ, AW)\mathsf{Rol}(X,Y)(A) - \mathsf{Rol}(Z,W)(A)R(X,Y) + \hat{R}(\mathsf{Rol}(Z,W)(A)X, AY)A \\ &+ \hat{R}(AX, \mathsf{Rol}(Z,W)(A)Y)A + \hat{R}(AX, AY)\mathsf{Rol}(Z,W)(A)\big)\big|_q.\end{aligned}$$

Proof. We use Proposition 3.37 where for U, V we take $U(A) = \mathsf{Rol}(X,Y)(A)$ and $V(A) = \mathsf{Rol}(Z,W)(A)$. First compute for B such that $\nu(B)|_q \in V|_q(Q)$ that

$$\begin{aligned}\nu(B)|_q U =& \nu(B)|_q\big(\tilde{A} \mapsto \tilde{A}R(X,Y) - \hat{R}(\tilde{A}X, \tilde{A}Y)\tilde{A}\big) \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0\big((A + tB)R(X,Y) - \hat{R}((A + tB)X, (A + tB)Y)(A + tB)\big) \\ =& BR(X,Y) - \hat{R}(BX, AY)A - \hat{R}(AX, BY)A - \hat{R}(AX, AY)B.\end{aligned}$$

So by taking $B = V(A)$ we get

$$\begin{aligned}\nu(V(A))|_q U =& \mathsf{Rol}(Z,W)(A)R(X,Y) - \hat{R}(\mathsf{Rol}(Z,W)(A)X, AY)A \\ &- \hat{R}(AX, \mathsf{Rol}(Z,W)(A)Y)A - \hat{R}(AX, AY)\mathsf{Rol}(Z,W)(A),\end{aligned}$$

and similarly for $\nu(U(A))|_q V$. $\square$

For later use, we find it convenient to provide another expression for Proposition 4.13 and, for that purpose, we recall the following notation. For $A, B \in \mathfrak{so}(T|_x M)$, we define

$$[A, B]_{\mathfrak{so}} := A \circ B - B \circ A \in \mathfrak{so}(T|_x M).$$

Then, one has the following corollary.

Corollary 4.14 For any $q = (x, \hat{x}; A) \in Q$ and $X, Y, Z, W \in \mathrm{VF}(M)$ we have

$$\begin{aligned}
&\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X,Y)(\cdot)), \nu(\mathsf{Rol}(Z,W)(\cdot))\big]\big|_q \\
=&A\big[R(X,Y), R(Z,W)\big]_{\mathfrak{so}} - \big[\hat{R}(AX,AY), \hat{R}(AZ,AW)\big]_{\mathfrak{so}}A \\
&- \hat{R}(\mathsf{Rol}(X,Y)(A)Z, AW)A - \hat{R}(AZ, \mathsf{Rol}(X,Y)(A)W)A \\
&+ \hat{R}(AX, \mathsf{Rol}(Z,W)(A)Y)A + \hat{R}(\mathsf{Rol}(Z,W)(A)X, AY)A.
\end{aligned} \tag{33}$$

4.3 Controllability Properties of $\mathcal{D}_{\mathrm{R}}$ and first results

Proposition 4.8 has the following simple consequence.

Corollary 4.15 The following cases are equivalent:

(i) The rolling distribution $\mathcal{D}_{\mathrm{R}}$ on Q is involutive.

(ii) For all $X, Y, Z \in T|_x M$ and $(x, \hat{x}; A) \in T^*(M) \otimes T(\hat{M})$

$$\mathsf{Rol}(X,Y)(A) = 0.$$

(iii) (M, g) and $(\hat{M}, \hat{g})$ both have constant and equal curvature.

The same result holds when one replaces Q by $T^*M \otimes T\hat{M}$.

Proof. (i) $\iff$ (ii) follows from Proposition 4.8.

(ii)$\Rightarrow$(iii) We use

$$\sigma_{(X,Y)} = g(R(X,Y)Y, X), \text{ and } \sigma_{(\hat{X},\hat{Y})} = \hat{g}(\hat{R}(\hat{X},\hat{Y})\hat{Y}, \hat{X}),$$

to denote the sectional curvature of M w.r.t orthonormal vectors $X, Y \in T|_x M$ and the sectional curvature of $\hat{M}$ w.r.t. orthonormal vectors $\hat{X}, \hat{Y} \in T|_{\hat{x}}\hat{M}$ respectively. The assumption that $\mathsf{Rol} = 0$ on Q then implies

$$\sigma_{(X,Y)} = \hat{\sigma}_{(AX,AY)}, \quad \forall (x, \hat{x}; A) \in Q,\ X, Y \in T|_x M. \tag{34}$$

If we fix $x \in M$ and g-orthonormal vectors $X, Y \in T|_x M$, then, for any $\hat{x} \in \hat{M}$ and any $\hat{g}$-orthonormal vectors $\hat{X}, \hat{Y} \in T|_{\hat{x}}\hat{M}$, we may choose $A \in Q|_{(x,\hat{x})}$ such that $AX = \hat{X}$, $AY = \hat{Y}$ (in the case $n = 2$ we may have to replace, say, $\hat{X}$ by $-\hat{X}$ but this does not change anything in the argument below). Hence the above equation (34) shows that the sectional curvatures at every point $\hat{x} \in \hat{M}$ and w.r.t every orthonormal pair $\hat{X}, \hat{Y}$ are all the same i.e., $\sigma_{(X,Y)}$. Thus $(\hat{M}, \hat{g})$ has constant sectional curvatures i.e., it has a constant curvature. Changing the roles of M and

$\hat{M}$ we see that (M, g) also has constant curvature and the constants of curvatures are the same.

(iii)⇒(ii) Suppose that $M, \hat{M}$ have constant and equal curvatures. By a standard result (see [30] Lemma II.3.3), this is equivalent to the fact that there exists $k \in \mathbb{R}$ such that

$$R(X,Y)Z = k\big(g(Y,Z)X - g(X,Z)Y\big), \quad X,Y,Z \in T|_x M,\ x \in M,$$
$$\hat{R}(\hat{X},\hat{Y})\hat{Z} = k\big(\hat{g}(\hat{Y},\hat{Z})\hat{X} - \hat{g}(\hat{X},\hat{Z})\hat{Y}\big), \quad \hat{X},\hat{Y},\hat{Z} \in T|_{\hat{x}}\hat{M},\ \hat{x} \in \hat{M}.$$

On the other hand, if $A \in Q$, $X, Y, Z \in T|_x M$, we would then have

$$\begin{aligned} &\hat{R}(AX,AY)(AZ) = k(\hat{g}(AY,AZ)AX - \hat{g}(AX,AZ)(AY)) \\ =&A(k(g(Y,Z)X - g(X,Z)Y) = A(R(X,Y)Z). \end{aligned}$$

This implies that $\mathsf{Rol}(X,Y)(A) = 0$ since Z was arbitrary.

□

In the situation of the previous corollary, the control system $(\Sigma)_R$ is as far away from being controllable as possible: all the orbits $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$, $q \in Q$, are integral manifolds of $\mathcal{D}_{\mathrm{R}}$. The next consequence of Proposition 4.8 can be seen as a (partial) generalization of the previous corollary and a special case of the Ambrose's theorem. The corollary gives a necessary and sufficient condition describing the case in which at least one $\mathcal{D}_{\mathrm{R}}$-orbit is an integral manifold of $\mathcal{D}_{\mathrm{R}}$.

Corollary 4.16 Suppose that (M, g) and $(\hat{M}, \hat{g})$ are complete. The following cases are equivalent:

(i) There exists a $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ such that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold of $\mathcal{D}_{\mathrm{R}}$.

(ii) There exists a $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ such that

$$\mathsf{Rol}(X,Y)(A) = 0, \quad \forall (x,\hat{x};A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\ X,Y \in T|_x M.$$

(iii) There is a complete Riemannian manifold (N, h) and Riemannian covering maps $F : N \to M$, $G : N \to \hat{M}$. In particular, (M, g) and $(\hat{M}, \hat{g})$ are locally isometric.

Proof. (i) ⇒ (ii): Notice that the restrictions of vector fields $\mathscr{L}_{\mathrm{R}}(X)$, $X \in \mathrm{VF}(M)$, to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ are smooth vector fields of that orbit. Thus $[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]$ is also tangent to this orbit for any $X, Y \in \mathrm{VF}(M)$ and hence Proposition 4.8 implies the claim.

(ii) ⇒ (i): It follows, from Proposition 4.8, that $\mathcal{D}_{\mathrm{R}}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}$, the restriction of $\mathcal{D}_{\mathrm{R}}$ to the manifold $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, is involutive. Since maximal connected integral manifolds of an involutive distribution are exactly its orbits, it follows that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold of $\mathcal{D}_{\mathrm{R}}$.

(i) ⇒ (iii): Let $N := \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and $h := (\pi_{Q,M}|_N)^*(g)$ i.e., for $q = (x, \hat{x}; A) \in N$ and $X, Y \in T|_x M$, define

$$h(\mathscr{L}_{\mathrm{R}}(X)|_q, \mathscr{L}_{\mathrm{R}}(Y)|_q) = g(X,Y).$$

If $F := \pi_{Q,M}|_N$ and $G := \pi_{Q,\hat{M}}|_N$, we immediately see that F is a local isometry (note that $\dim(N) = n$) and the fact that G is a local isometry follows from the following computation: for $q = (x, \hat{x}; A) \in N$, $X, Y \in T|_x M$, one has

$$\hat{g}(G_*(\mathscr{L}_{\mathrm{R}}(X)|_q), G_*(\mathscr{L}_{\mathrm{R}}(Y)|_q)) = \hat{g}(AX, AY) = g(X, Y) = h(\mathscr{L}_{\mathrm{R}}(X)|_q, \mathscr{L}_{\mathrm{R}}(Y)|_q).$$

The completeness of (N, h) can be easily deduced from the completeness of M and $\hat{M}$ together with Proposition 3.20. Proposition II.1.1 in [30] proves that the maps F, G are in fact (surjective and) Riemannian coverings.

(iii) $\Rightarrow$ (ii): Let $x_0 \in M$ and choose $z_0 \in N$ such that $F(z_0) = x_0$. Define $\hat{x}_0 = G(z_0) \in \hat{M}$ and $A_0 := G_*|_{z_0} \circ (F_*|_{z_0})^{-1}$ which is an element of $Q|_{(x_0,\hat{x}_0)}$ since F, G were local isometries. Write $q_0 = (x_0, \hat{x}_0; A_0) \in Q$.

Let $\gamma : [0, 1] \to M$ be an a.c. curve with $\gamma(0) = x_0$. Since F is a smooth covering map, there is a unique a.c. curve $\Gamma : [0, 1] \to N$ with $\gamma = F \circ \Gamma$ and $\Gamma(0) = z_0$. Define $\hat{\gamma} = G \circ \Gamma$ and $A(t) = G_*|_{\Gamma(t)} \circ (F_*|_{\Gamma(t)})^{-1} \in Q$, $t \in [0, 1]$. It follows that, for a.e. $t \in [0, 1]$,

$$\dot{\hat{\gamma}}(t) = G_*|_{\Gamma(t)}\dot{\Gamma}(t) = A(t)\dot{\gamma}(t).$$

Since F, G are local isometries, $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}A(\cdot) = 0$ for a.e. $t \in [0, 1]$. Thus $t \mapsto (\gamma(t), \hat{\gamma}(t); A(t))$ is the unique rolling curve along γ starting at q_0 and defined on $[0, 1]$ and therefore curves of Q formed in this fashion fill up the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Moreover, since F, G are local isometries, it follows that for every $z \in N$ and $X, Y \in T|_{F(z)}M$, $\mathsf{Rol}(X, Y)(G_*|_z \circ (F_*|_z)^{-1}) = 0$. These facts prove that the condition in (ii) holds and the proof is therefore finished. $\square$

Remark 4.17 If one does not assume that (M, g) and $(\hat{M}, \hat{g})$ are complete in Corollary 4.16, then (iii) in the above corollary must be replaced by the following:

(iii)' There is a connected Riemannian manifold (N, h) (not necessarily complete) and Riemannian covering maps $F : N \to M^\circ$, $G : N \to \hat{M}^\circ$ where M°, $\hat{M}^\circ$ are open sets of M and $\hat{M}$ and there is a $z_0 \in N$ such that if $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ is defined by $A_0 := G_*|_{z_0} \circ (F_*|_{z_0})^{-1}$, then $M^\circ = \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$ and $\hat{M}^\circ = \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$.

In particular, M°, $\hat{M}^\circ$ are connected and (M°, g), $(\hat{M}^\circ, \hat{g})$ are locally isometric. Indeed, the argument in the implication (i) $\Rightarrow$ (iii) goes through except for the completeness of (N, h), where $N = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ (connected). Proposition 4.2 and Remark 4.2 show that $F = \pi_{Q,M}|_N : N \to M^\circ$, $G = \pi_{Q,\hat{M}}|_N : N \to \hat{M}^\circ$ are bundles with discrete fibers. Now it is a standard (easy) fact that a bundle $\pi : X \to Y$ with connected total space X and discrete fibers is a covering map (this could have been used in the above proof instead of referring to [30]). On the other hand, in the argument of the implication (iii) $\Rightarrow$ (ii) we did not even use completeness of (N, h) but only the fact that $F : N \to M$ is a covering map to lift a curve γ in M to the curve Γ in Q. In this non-complete setting, we just have to consider using curves γ in M° and lift them to N by using $F : N \to M^\circ$. Indeed, if $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, there is a curve $\gamma : [0, 1] \to M$ such that $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1) = q$. For all t one has

$$\gamma(t) = \pi_{Q,M}(q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)) \in \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) = M^\circ,$$

so γ is actually a curve in M°.

Finally, notice that the assumption in (iii)' that $\hat{M}^\circ = \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$ follows from the others. Indeed, making only the other assumptions, it is first of all clear that if q and γ are as above, then

$$\pi_{Q,\hat{M}}(q) = \pi_{Q,\hat{M}}(q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1)) = G(\Gamma(1)) \in \hat{M}^\circ,$$

so $\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \subset \hat{M}^\circ$. Then if $\hat{x} \in \hat{M}^\circ$, one may take a path $\hat{\gamma} : [0,1] \to \hat{M}^\circ$ such that $\hat{\gamma}(0) = \hat{x}_0$, $\hat{\gamma}(1) = \hat{x}$ and lift it by the covering map G to a curve $\hat{\Gamma}(t)$ in N starting from z_0. Then if $\gamma(t) := F(\hat{\Gamma}(t))$, $t \in [0,1]$, we easily see that $\hat{\gamma} = \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)$, whence $\hat{x} = \hat{\gamma}(1) \in \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$.

On the opposite direction with respect to having the rolling curvature equal to zero, one gets the following proposition.

Proposition 4.18 Suppose there is a point $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and $\epsilon > 0$ such that for every $X \in \mathrm{VF}(M)$ with $\|X\|_g < \epsilon$ on M one has

$$V|_{\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t,q_0)}(\pi_Q) \subset T\big(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\big), \quad |t| < \epsilon.$$

Then the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is open in Q. As a consequence, we have the following characterization of complete controllability: the control system $(\Sigma)_R$ is completely controllable if and only if

$$\forall q \in Q, \quad V|_q(\pi_Q) \subset T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q). \tag{35}$$

Proof. For the first part of the proposition, the assumptions and Lemma 4.20 given below imply that for every $X \in T|_{x_0}M$ we have $\mathscr{L}_{\mathrm{NS}}(Y, \hat{Y})|_{q_0} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for every $Y \in X^\perp$, $\hat{Y} \in A_0 X^\perp$. But since X is an arbitrary element of $T|_{x_0}M$, this means that $\mathcal{D}_{\mathrm{NS}}|_{q_0} \subset T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and because $T|_{q_0}Q = \mathcal{D}_{\mathrm{NS}}|_{q_0} \oplus V|_{q_0}(\pi_Q)$, we get $T|_{q_0}Q = T|_{q_0}\big(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\big)$. This implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is open in Q. The last part of the proposition is an immediate consequence of this and the fact that Q is connected. $\square$

Remark 4.19 The above corollary is intuitively obvious. Assumption given by Eq. (35) simply means that there is complete freedom for infinitesimal spinning, i.e., for reorienting one manifold with respect to the other one without moving in $M \times \hat{M}$. In that case, proving complete controllability is easy, by using a crab-like motion.

We end this section by providing a technical lemma needed for the argument of the previous proposition. It is actually a consequence of Proposition 3.36.

Lemma 4.20 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. Suppose that, for some $X \in \mathrm{VF}(M)$ and a real sequence $(t_n)_{n=1}^\infty$ s.t. $t_n \neq 0$ for all n, $\lim_{n\to\infty} t_n = 0$, we have, for every $n \geq 0$,

$$V|_{\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t_n,q_0)}(\pi_Q) \subset T(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)). \tag{36}$$

Then $\mathscr{L}_{\mathrm{NS}}(Y, \hat{Y})\big|_{q_0} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for every $Y \in T|_{x_0}M$ that is g-orthogonal to $X|_{x_0}$ and every $\hat{Y} \in T|_{\hat{x}_0}\hat{M}$ that is $\hat{g}$-orthogonal to $A_0X|_{x_0}$. Hence the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ has codimension at most 1 inside Q.

Proof. Letting n tend to infinity, it follows from (36) that $V|_{q_0}(\pi_Q) \subset T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Recall, from Proposition 3.4, that every element of $V|_{q_0}(\pi_Q)$ is of the form $\nu(B)|_{q_0}$, with a unique $B \in A_0\mathfrak{so}(T|_{x_0}M)$. Fix such a B and define a smooth local section $\tilde{S}$ of $\mathfrak{so}(TM) \to M$ defined on an open set $W \ni x_0$ by

$$\tilde{S}|_x = P_0^1\big(t \mapsto \exp_{x_0}(t\exp_{x_0}^{-1}(x))\big)(A_0^{\overline{T}}B).$$

Then clearly, $\tilde{S}|_{x_0} = A_0^{\overline{T}}B$ and $\nabla_Y\tilde{S} = 0$ for all $Y \in T|_{x_0}M$ and it is easy to verify that $\tilde{S}|_x \in \mathfrak{so}(T|_xM)$ for all $x \in W$. We next define a smooth map $U : \pi_Q^{-1}(W \times \hat{M}) \to T^*M \otimes T\hat{M}$ by $U(x, \hat{x}; A) = A\tilde{S}|_x$. Obviously $\nu(U(x, \hat{x}; A)) \in V|_{(x,\hat{x};A)}(\pi_Q)$ for all $(x, \hat{x}; A)$. Then, choosing in Proposition 3.36, $\overline{T} = X + (\cdot)X$ (and the above U) and noticing that

$$\nu(U(A_0))|_{q_0}\overline{T} = U(A_0)X = BX,$$

one gets

$$[\mathscr{L}_{\mathrm{R}}(X), \nu(U(\cdot))]|_{q_0} = -\mathscr{L}_{\mathrm{NS}}(BX)|_{q_0} + \nu(\overline{\nabla}_{(X,A_0X)}(U(\tilde{A})))|_{q_0} \tag{37}$$

where $\tilde{A}|_{(x_0,\hat{x}_0)} = A_0$. By the choice of $\tilde{S}$ and $\tilde{A}$, we have, for all $\overline{Y} = (Y, \hat{Y}) \in T|_{(x_0,\hat{x}_0)}M \times \hat{M}$,

$$\nabla_{\overline{Y}}(U(\tilde{A})) = \nabla_{\overline{Y}}(\tilde{A}\tilde{S}) = (\nabla_{\overline{Y}}\tilde{A})\tilde{S}|_{(x_0,\hat{x}_0)} + \tilde{A}|_{(x_0,\hat{x}_0)}\nabla_Y\tilde{S} = 0,$$

and hence the last term on the right hand side of (37) actually vanishes.

By definition, the vector field $q \mapsto \mathscr{L}_{\mathrm{R}}(X)|_q$ is tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and, by the assumption of Equation (36), the values of the map $q = (x, \hat{x}; A) \mapsto \nu(U(A))|_q$ are also tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ at the points $\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t_n, q_0)$, $n \in \mathbb{N}$. Hence $\big((\Phi_{\mathscr{L}_{\mathrm{R}}(X)})_{-t_n}\big)_*\nu(U(\cdot))|_{\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t_n,q_0)} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and therefore,

$$\begin{aligned}&[\mathscr{L}_{\mathrm{R}}(X), \nu(U(\cdot))]|_{q_0}\\ &= \lim_{n\to\infty} \frac{\big((\Phi_{\mathscr{L}_{\mathrm{R}}(X)})_{-t_n}\big)_*\nu(U(\cdot))|_{\Phi_{\mathscr{L}_{\mathrm{R}}(X)}(t_n,q_0)} - \nu(B)|_{q_0}}{t_n} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\end{aligned}$$

i.e., the left hand side of (37) must belong to $T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. But this implies that

$$\mathscr{L}_{\mathrm{NS}}(BX)|_{q_0} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0), \quad \forall B \text{ s.t. } \nu(B) \in V|_{q_0}(\pi_Q)$$

i.e.,

$$\mathscr{L}_{\mathrm{NS}}(A_0\mathfrak{so}(T|_{x_0}M)X)|_{q_0} \subset T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

Notice next that $\mathfrak{so}(T|_{x_0}M)X$ is exactly the set $X|_{x_0}^{\perp}$ of vectors of $T|_{x_0}M$ that are g-perpendicular to $X|_{x_0}$. Since $A_0 \in Q$, it follows that the set $A_0\mathfrak{so}(T|_{x_0}M)X$ is equal to $A_0X|_{x_0}^{\perp}$ which is the set of vectors of $T|_{\hat{x}_0}\hat{M}$ that are $\hat{g}$-perpendicular to $A_0X|_{x_0}$. We conclude that $\mathscr{L}_{\mathrm{NS}}(Y)|_{q_0} = \mathscr{L}_{\mathrm{R}}(Y)|_{q_0} - \mathscr{L}_{\mathrm{NS}}(A_0Y)|_{q_0} \in T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $Y \in X|_{x_0}^{\perp}$.

Finally notice that since the subspaces $X^{\perp} \times \{0\}$, $\mathbb{R}(X, A_0X)$ and $\{0\} \times (A_0X)^{\perp}$ of $T|_{(x_0,\hat{x}_0)}(M \times \hat{M})$ are linearly independent, their $\mathscr{L}_{\mathrm{NS}}$-lifts at q_0 are that also and hence these lifts span a $(n-1)+1+(n-1) = 2n-1$ dimensional subspace of $T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. This combined with the fact that $V|_{q_0}(\pi_Q) \subset T|_{q_0}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ shows $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 2n-1+\dim V|_{q_0}(\pi_Q) = \dim(Q)-1$ i.e., the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ has codimension at most 1 in Q. This finishes the proof.

□

5 Rolling Problem (R) in 3D

As mentioned in introduction, the goal of this chapter is to provide a local structure theorem of the orbits $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ when M and $\hat{M}$ are 3-dimensional Riemannian manifolds. Recall that complete controllability of $(\Sigma)_R$ is equivalent to openess of *all* the orbits of $(\Sigma)_R$, thanks to the fact that Q is connected and $(\Sigma)_R$ is driftless. In case there is no complete controllability, then there exists a non open orbit which is an immersed manifold in Q of dimension at most eigth. Moreover, as a fiber bundle over M, the fiber has dimension at most five.

5.1 Statement of the Results and Proof Strategy

Our first theorem provides all the possibilities for the local structure of a non open orbit for the rolling (R) of two 3D Riemannian manifolds.

Theorem 5.1 Let (M, g), $(\hat{M}, \hat{g})$ be 3-dimensional Riemannian manifolds. Assume that $(\Sigma)_R$ is not completely controllable and let $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, for some $q_0 \in Q$, be a non open orbit. Then, there exists an open and dense subset O of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ so that, for every $q_1 = (x_1, \hat{x}_1; A_1) \in O$, there are neighbourhoods U of x_1 and $\hat{U}$ of $\hat{x}_1$ such that one of the following holds:

(a) $(U, g|_U)$ and $(\hat{U}, \hat{g}|_{\hat{U}})$ are (locally) isometric;

(b) $(U, g|_U)$ and $(\hat{U}, \hat{g}|_{\hat{U}})$ are both of class $\mathcal{M}_\beta$ for some $\beta > 0$;

(c) $(U, g|_U)$ and $(\hat{U}, \hat{g}|_{\hat{U}})$ are both isometric to warped products $(I \times N, h_f)$, $(I \times \hat{N}, \hat{h}_{\hat{f}})$ for some open interval $I \subset \mathbb{R}$ and warping functions $f, \hat{f}$ which moreover satisfy either

(A) $\dfrac{f'(t)}{f(t)} = \dfrac{\hat{f}'(t)}{\hat{f}(t)}$ for all $t \in I$ or

(B) there is a constant $K \in \mathbb{R}$ such that $\dfrac{f''(t)}{f(t)} = -K = \dfrac{\hat{f}''(t)}{\hat{f}(t)}$ for all $t \in I$.

For the definition and results on warped products and class $\mathcal{M}_\beta$, we refer to Appendix C.2.

Note that we do not address here to the issue of the global structure of a non open orbit for the rolling (R) of two 3D Riemmanian manifolds. For that, one would have to "glue" together the local information provided by Theorem 5.1. Instead, our second theorem below shows, in some sense, that the list of possibilities established in Theorem 5.1 is complete. We will exclude the case where $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold since in this case this orbit has dimension 3 and (M, g), $(\hat{M}, \hat{g})$ are locally isometric, see Corollary 4.16 and Remark 4.17.

Theorem 5.2 Let (M, g), $(\hat{M}, \hat{g})$ be 3D Riemannian manifolds, $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and suppose $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$. If one writes $M^\circ := \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $\hat{M}^\circ := \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, then the following holds true.

(a) If (M,g), $(\hat{M},\hat{g})$ are both of class $\mathcal{M}_\beta$ and if E_1,E_2,E_3 and $\hat{E}_1,\hat{E}_2,\hat{E}_3$ are adapted frames of (M,g) and $(\hat{M},\hat{g})$, respectively, then one has:

   (A) If $A_0E_2|_{x_0}=\pm\hat{E}_2|_{\hat{x}_0}$, then $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=7$;

   (B) If $A_0E_2|_{x_0}\neq\pm\hat{E}_2|_{\hat{x}_0}$ and if (only) one of (M°,g) or $(\hat{M}^\circ,\hat{g})$ has constant curvature, then $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=7$;

   (C) Otherwise, $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=8$.

(b) If $(M,g)=(I\times N,h_f)$, $(\hat{M},\hat{g})=(\hat{I}\times\hat{N},\hat{h}_{\hat{f}})$ are warped products, where $I,\hat{I}\subset\mathbb{R}$ are open intervals, and if $x_0=(r_0,y_0)$, $\hat{x}_0=(\hat{r}_0,\hat{y}_0)$, then one has

   (A) If $A_0\frac{\partial}{\partial r}|_{(r_0,y_0)}=\frac{\partial}{\partial r}|_{(\hat{r}_0,\hat{y}_0)}$ and if for every t s.t. $(t+r_0,t+\hat{r}_0)\in I\times\hat{I}$ it holds

$$\frac{f'(t+r_0)}{f(t+r_0)}=\frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)},$$

   then $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=6$;

   (B) Suppose there is a constant $K\in\mathbb{R}$ such that $\frac{f''(r)}{f(r)}=-K=\frac{\hat{f}''(\hat{r})}{\hat{f}(\hat{r})}$ for all $(r,\hat{r})\in I\times\hat{I}$.

   (B1) If $A_0\frac{\partial}{\partial r}|_{(r_0,y_0)}=\pm\frac{\partial}{\partial r}|_{(\hat{r}_0,\hat{y}_0)}$ and $\frac{f'(r_0)}{f(r_0)}=\pm\frac{\hat{f}'(\hat{r}_0)}{\hat{f}(\hat{r}_0)}$, with $\pm$-cases correspondingly on both cases, then $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=6$;

   (B2) If (only) one of (M°,g), $(\hat{M}^\circ,\hat{g})$ has constant curvature, then one has $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=6$;

   (B3) Otherwise $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=8$.

Here $(r,y)\mapsto\frac{\partial}{\partial r}|_{(r,y)}$, $(\hat{r},\hat{y})\mapsto\frac{\partial}{\partial r}|_{(\hat{r},\hat{y})}$, are the vector fields in $I\times N$ and $\hat{I}\times\hat{N}$ induced by the canonical, positively oriented vector field $r\mapsto\frac{\partial}{\partial r}\big|_r$ on $I,\hat{I}\subset\mathbb{R}$.

From now on(M,g), $(\hat{M},\hat{g})$ will be connected, oriented 3-dimensional Riemannian manifolds. The Hodge-duals of (M,g), $(\hat{M},\hat{g})$ are denoted by $\star:=\star_M$ and $\hat{\star}:=\star_{\hat{M}}$, respectively.

As a reminder, for $q_0=(x_0,\hat{x}_0;A_0)\in Q$, we will write

$$\begin{aligned}
\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}:=&\pi_Q|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}:\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\to M\times\hat{M},\\
\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}:=&\mathrm{pr}_1\circ\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}:\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\to M,\\
\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}}:=&\mathrm{pr}_2\circ\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}:\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\to\hat{M},
\end{aligned}$$

where $\mathrm{pr}_1:M\times\hat{M}\to M$, $\mathrm{pr}_2:M\times\hat{M}\to\hat{M}$ are projections onto the first and second factor, respectively. Before we start the arguments for Theorems 5.1 and 5.2, we give next two propositions which are both instrumental in these arguments and also of independant interest.

Proposition 5.3 Let (M,g), $(\hat{M},\hat{g})$ be two Riemannian manifolds of dimension 3, $q_0=(x_0,\hat{x}_0;A_0)\in Q$ and suppose there is an open subset O of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and a smooth unit vector field $E_2\in \mathrm{VF}(\pi_{Q,M}(O))$ such that $\nu(A\star E_2)|_q$ is tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $q\in O$. If the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open in Q, then for any $x\in\pi_{Q,M}(O)$ and any unit vector fields E_1,E_3 such that E_1,E_2,E_3 is an orthonormal frame in some neighbourhood U of x in M, the connection table associated to E_1,E_2,E_3 is given by

$$\Gamma=\begin{pmatrix}\Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)}\\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)}\\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)}\end{pmatrix},$$

and

$$V(\Gamma^1_{(2,3)})=0,\quad V(\Gamma^1_{(1,2)})=0,\quad \forall V\in E_2|_y^{\perp},\quad y\in U,$$

where $\Gamma=[(\Gamma^j_{\star i})^i_j]$, $\Gamma^j_{(i,k)}=g(\nabla_{E_j}E_i,E_k)$ and $\star 1=(2,3)$, $\star 2=(3,1)$ and $\star 3=(1,2)$.

Remark 5.4 In particular, this means that the assumptions of the previous proposition imply that the assumptions of Proposition C.17 are fulfilled.

Proof. Notice that $\pi_{Q,M}(O)$ is open in M since $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}=\pi_{Q,M}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}$ is a submersion. Without loss of generality, we may assume that there exist $E_1,E_3\in \mathrm{VF}(\pi_{Q,M}(O))$ such that E_1,E_2,E_3 form an orthonormal basis.

We begin by computing in O the following Lie bracket,

$$\begin{aligned}[\mathscr{L}_{\mathrm{R}}(E_2),\nu((\cdot)\star E_2)]|_q=&-\mathscr{L}_{\mathrm{NS}}(A(\star E_2)E_2)|_q+\nu(A\star\nabla_{E_2}E_2)|_q\\ =&\nu(A\star(-\Gamma^2_{(1,2)}E_1+\Gamma^2_{(2,3)}E_3))|_q=:V_2|_q,\end{aligned}$$

whence V_2 is a vector field in O and furthermore

$$\begin{aligned}[V_2,\nu((\cdot)\star E_2)]|_q=&\nu(A[\star(-\Gamma^2_{(1,2)}E_1+\Gamma^2_{(2,3)}E_3),\star E_2]_{\mathfrak{so}})|_q\\ =&\nu(A\star(-\Gamma^2_{(1,2)}E_3-\Gamma^2_{(2,3)}E_1))|_q=:M_2|_q,\end{aligned}$$

where M_2 is a vector field in O as well. Now if there were an open subset O' of O the $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}$-vertical vector fields where $\nu(A\star E_2)|_q,V_2|_q,M_2|_q$ were linearly independent for all $q\in O'$, it would follow that they form a basis of $V|_q(\pi_Q)$ for $q\in O'$ and hence $V|_q(\pi_Q)\subset T|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$ for $q\in O'$. Then Corollary 4.18 would imply that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is open, which is a contradiction. Hence in a dense subset O_d of O one has that $\nu(A\star E_2)|_q,V_2|_q,M_2|_q$ are linearly dependent which implies

$$0=\det\begin{pmatrix}0 & 1 & 0\\ -\Gamma^2_{(1,2)} & 0 & \Gamma^2_{(2,3)}\\ -\Gamma^2_{(2,3)} & 0 & -\Gamma^2_{(1,2)}\end{pmatrix}=-((\Gamma^2_{(1,2)})^2+(\Gamma^2_{(2,3)})^2),$$

i.e.,

$$\Gamma^2_{(1,2)}=0,\quad \Gamma^2_{(2,3)}=0,$$

on $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),M}(O_d)$. It is clear that $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),M}(O_d)$ is dense in $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),M}(O)$ so the above relation holds on the open subset $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),M}(O)$ of M.

Next compute

$$\begin{aligned}
[\mathscr{L}_{\rm R}(E_1),\nu((\cdot)\star E_2)]|_q =&\mathscr{L}_{\rm NS}(AE_3)|_q+\nu(A\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3))|_q=\mathscr{L}_{\rm R}(E_3)|_q-L_3|_q,\\
[\mathscr{L}_{\rm R}(E_3),\nu((\cdot)\star E_2)]|_q =&-\mathscr{L}_{\rm NS}(AE_1)|_q-\nu(A\star(-\Gamma^3_{(1,2)}E_1+\Gamma^3_{(2,3)}E_3))|_q\\
=&-\mathscr{L}_{\rm R}(E_1)|_q+L_1|_q,
\end{aligned}$$

where $L_1,L_3\in \mathrm{VF}(O')$ such that

$$\begin{aligned}
L_1|_q:=&\mathscr{L}_{\rm NS}(E_1)|_q+\nu(A\star(-\Gamma^3_{(1,2)}E_1+\Gamma^3_{(2,3)}E_3))|_q,\\
L_3|_q:=&\mathscr{L}_{\rm NS}(E_3)|_q-\nu(A\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3))|_q.
\end{aligned}$$

Continuing by taking brackets of these against $\nu(A\star E_2)|_q$ gives

$$\begin{aligned}
[L_1,\nu((\cdot)\star E_2)]|_q =&\nu(A\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3))|_q+\nu(A[\star(-\Gamma^3_{(1,2)}E_1+\Gamma^3_{(2,3)}E_3),\star E_2]_{\mathfrak{so}})|_q\\
=&\nu(A\star(-(\Gamma^1_{(1,2)}+\Gamma^3_{(2,3)})E_1+(\Gamma^1_{(2,3)}-\Gamma^3_{(1,2)})E_3)|_q=:M_3,\\
[L_3,\nu((\cdot)\star E_2)]|_q =&\nu(A\star(-\Gamma^3_{(1,2)}E_1+\Gamma^3_{(2,3)}E_3))|_q-\nu(A[\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3),\star E_2]_{\mathfrak{so}})|_q\\
=&\nu(A\star((-\Gamma^3_{(1,2)}+\Gamma^1_{(2,3)})E_1+(\Gamma^3_{(2,3)}+\Gamma^1_{(1,2)})E_3)|_q=:M_1.
\end{aligned}$$

Since $\nu(A\star E_2)|_q,M_1|_q,M_3|_q$ are smooth $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)}$-vertical vector fields defined on O', we may again resort to Corollary 4.18 to deduce that

$$0=\det\begin{pmatrix}0 & 1 & 0\\ -(\Gamma^1_{(1,2)}+\Gamma^3_{(2,3)}) & 0 & \Gamma^1_{(2,3)}-\Gamma^3_{(1,2)}\\ -\Gamma^3_{(1,2)}+\Gamma^1_{(2,3)} & 0 & \Gamma^3_{(2,3)}+\Gamma^1_{(1,2)}\end{pmatrix}=-((\Gamma^1_{(1,2)}+\Gamma^3_{(2,3)})^2+(\Gamma^1_{(2,3)}-\Gamma^3_{(1,2)})^2),$$

i.e., $\Gamma^3_{(2,3)}=-\Gamma^1_{(1,2)}$, $\Gamma^3_{(1,2)}=\Gamma^1_{(2,3)}$ on $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),M}(O)$. We will now prove that derivatives of $\Gamma^1_{(2,3)}$ and $\Gamma^1_{(1,2)}$ in the $E_2^\perp$-directions vanish on $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),M}(O)$. To reach this we first notice that

$$L_1|_q=\mathscr{L}_{\rm NS}(E_1)|_q-\nu(A\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3))|_q,$$

and then compute

$$\begin{aligned}
[\mathscr{L}_{\rm R}(E_1),L_1]|_q =&\mathscr{L}_{\rm NS}(\Gamma^1_{(1,2)}E_2-\Gamma^1_{(3,1)}E_3)|_q-\mathscr{L}_{\rm R}(\nabla_{E_1}E_1)|_q\\
&+\nu(AR(E_1\wedge E_1)-\hat{R}(AE_1\wedge 0)A)|_q\\
&+\Gamma^1_{(1,2)}\mathscr{L}_{\rm NS}(AE_2)|_q-\nu\big(A\star(E_1(\Gamma^1_{(2,3)})E_1+E_1(\Gamma^1_{(1,2)})E_3)\big)|_q\\
&-\nu\big(A\star(\Gamma^1_{(2,3)}(\Gamma^1_{(1,2)}E_2-\Gamma^1_{(3,1)}E_3)+\Gamma^1_{(1,2)}(\Gamma^1_{(3,1)}E_1-\Gamma^1_{(2,3)}E_2))\big)|_q\\
=&\Gamma^1_{(1,2)}\mathscr{L}_{\rm R}(E_2)|_q-\Gamma^1_{(3,1)}L_3|_q-\mathscr{L}_{\rm R}(\nabla_{E_1}E_1)|_q\\
&-\nu\big(A\star(E_1(\Gamma^1_{(2,3)})E_1+E_1(\Gamma^1_{(1,2)})E_3)\big)|_q.
\end{aligned}$$

So if one define $J_1|_q:=\nu\big(A\star(E_1(\Gamma^1_{(2,3)})E_1+E_1(\Gamma^1_{(1,2)})E_3)\big)|_q$, then J_1 is a smooth vector field in O (tangent to $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$) and

$$[J_1,\nu((\cdot)\star E_2]|_q=\nu\big(A\star(E_1(\Gamma^1_{(2,3)})E_3-E_1(\Gamma^1_{(1,2)})E_1)\big)|_q.$$

Since $\nu(A\star E_1)|_q, J_1|_q$ and $[J_1,\nu((\cdot)\star E_2]|_q$ are $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)}$ vertical vector fields in O and $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ is not open, we again deduce that

$$E_1(\Gamma^1_{(2,3)})=0,\quad E_1(\Gamma^1_{(1,2)})=0.$$

In a similar way,

$$\begin{aligned}
[\mathscr{L}_{\rm R}(E_3),L_3]|_q =&\mathscr{L}_{\rm NS}(\Gamma^3_{(3,1)}E_1+\Gamma^1_{(1,2)}E_2)|_q-\mathscr{L}_{\rm R}(\nabla_{E_3}E_3)|_q\\
&+\nu(AR(E_3\wedge E_3)-\hat{R}(AE_3\wedge 0)A)|_q\\
&+\Gamma^1_{(1,2)}\mathscr{L}_{\rm NS}(AE_2)|_q-\nu\big(A\star(-E_3(\Gamma^1_{(1,2)})E_1+E_3(\Gamma^1_{(2,3)})E_3)\big)|_q\\
&-\nu\big(A\star(-\Gamma^1_{(1,2)}(\Gamma^1_{(2,3)}E_2-\Gamma^3_{(3,1)}E_3)+\Gamma^1_{(2,3)}(\Gamma^3_{(3,1)}E_1+\Gamma^1_{(1,2)}E_2)\big)|_q\\
=&\Gamma^3_{(3,1)}L_1|_q+\Gamma^1_{(1,2)}\mathscr{L}_{\rm R}(E_2)|_q-\mathscr{L}_{\rm R}(\nabla_{E_3}E_3)|_q\\
&-\nu\big(A\star(-E_3(\Gamma^1_{(1,2)})E_1+E_3(\Gamma^1_{(2,3)})E_3)\big)|_q,
\end{aligned}$$

so $J_3|_q:=\nu\big(A\star(-E_3(\Gamma^1_{(1,2)})E_1+E_3(\Gamma^1_{(2,3)})E_3)\big)|_q$ defines a smooth vector field on O and

$$[J_3,\nu((\cdot)\star E_2)]|_q=\nu\big(A\star(-E_3(\Gamma^1_{(1,2)})E_3-E_3(\Gamma^1_{(2,3)})E_1)\big)|_q.$$

The same argument as before implies that $E_3(\Gamma^1_{(1,2)})=0$, $E_3(\Gamma^1_{(2,3)})=0$. Since $E_2^\perp$ is spanned by E_1,E_3, the claim follows. This completes the proof. $\square$

We next provide a completary result to Proposition 5.3 which will be fundamental for the proof of Theorem 5.2.

Proposition 5.5 Let (M,g), $(\hat{M},\hat{g})$ be two Riemannian manifolds of dimension 3, $q_0=(x_0,\hat{x}_0;A_0)\in Q$. Assume that there is an open subset O of $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ and a smooth orthonormal local frame $E_1,E_2,E_3\in\mathrm{VF}(U)$ defined on the open subset $U:=\pi_{Q,M}(O)$ of M with respect to which the connection table has the form

$$\Gamma=\begin{pmatrix}\Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)}\\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)}\\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)}\end{pmatrix},$$

and that moreover

$$V(\Gamma^1_{(2,3)})=0,\quad V(\Gamma^1_{(1,2)})=0,\quad \forall V\in E_2|_y^\perp,\quad y\in U.$$

Define smooth vector fields L_1,L_2,L_3 on the open subset $\tilde{O}:=\pi^{-1}_{Q,M}(U)$ of Q by

$$\begin{aligned}
L_1|_q=&\mathscr{L}_{\rm NS}(E_1)|_q-\nu(A\star(\Gamma^1_{(2,3)}E_1+\Gamma^1_{(1,2)}E_3))|_q\\
L_2|_q=&\Gamma^1_{(2,3)}(x)\mathscr{L}_{\rm NS}(E_2)|_q\\
L_3|_q=&\mathscr{L}_{\rm NS}(E_3)|_q-\nu(A\star(-\Gamma^1_{(1,2)}E_1+\Gamma^1_{(2,3)}E_3))|_q.
\end{aligned}$$

Then we have the following:

(i) If $\nu(A\star E_2)|_q$ is tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ at every point $q\in O$, then the vectors

$$\mathscr{L}_{\rm R}(E_1)|_q,\ \mathscr{L}_{\rm R}(E_2)|_q,\ \mathscr{L}_{\rm R}(E_3)|_q,\ \nu(A\star E_2)|_q,\ L_1|_q,\ L_2|_q,\ L_3|_q,$$

are all tangent to $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)$ for every $q\in O$.

(ii) On $\tilde{O}$ we have the following Lie-bracket formulas

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(E_1), \nu((\cdot)\star E_2)]|_q &= \mathscr{L}_{\mathrm{R}}(E_3)|_q - L_3|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_2), \nu((\cdot)\star E_2)]|_q &= 0, \\
[\mathscr{L}_{\mathrm{R}}(E_3), \nu((\cdot)\star E_2)]|_q &= -\mathscr{L}_{\mathrm{R}}(E_1)|_q + L_1|_q, \\
[L_1, \nu((\cdot)\star E_2)]|_q &= 0, \\
[L_3, \nu((\cdot)\star E_2)]|_q &= 0, \\
[\mathscr{L}_{\mathrm{R}}(E_1), L_1]|_q &= -\Gamma^1_{(3,1)} L_3|_q + \Gamma^1_{(3,1)}\mathscr{L}_{\mathrm{R}}(E_3)|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_3), L_3]|_q &= \Gamma^3_{(3,1)} L_1|_q - \Gamma^3_{(3,1)}\mathscr{L}_{\mathrm{R}}(E_1)|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_2), L_1]|_q &= \Gamma^1_{(1,2)} L_1|_q - (\Gamma^1_{(2,3)} + \Gamma^2_{(3,1)}) L_3|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_2), L_3]|_q &= (\Gamma^1_{(2,3)} + \Gamma^2_{(3,1)}) L_1|_q + \Gamma^1_{(1,2)} L_3|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_3), L_1]|_q &= 2L_2|_q - \Gamma^3_{(3,1)} L_3|_q - \mathscr{L}_{\mathrm{R}}(\nabla_{E_1} E_3)|_q - \Gamma^1_{(2,3)}\mathscr{L}_{\mathrm{R}}(E_2)|_q, \\
&\quad - (K_2 + (\Gamma^1_{(2,3)})^2 + (\Gamma^1_{(1,2)})^2)\nu(A\star E_2)|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_1), L_3]|_q &= -2L_2|_q + \Gamma^1_{(3,1)} L_1|_q - \mathscr{L}_{\mathrm{R}}(\nabla_{E_3} E_1)|_q + \Gamma^1_{(3,1)}\mathscr{L}_{\mathrm{R}}(E_2)|_q, \\
&\quad + (K_2 + (\Gamma^1_{(1,2)})^2 + (\Gamma^1_{(2,3)})^2)\nu(A\star E_2)|_q, \\
[L_3, L_1]|_q &= 2L_2|_q - \Gamma^1_{(3,1)} L_1|_q - \Gamma^3_{(3,1)} L_3|_q, \\
&\quad - (K_2 + (\Gamma^1_{(2,3)})^2 + (\Gamma^1_{(1,2)})^2)\nu(A\star E_2)|_q.
\end{aligned}$$

Proof. It has been already shown in the course of the proof of Proposition 5.3 that the vectors $\mathscr{L}_{\mathrm{R}}(E_1)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(E_3)|_q, \nu(A\star E_2)|_q, L_1|_q, L_3|_q$ are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for $q\in O$. Moreover, the first 7 brackets appearing in the statement of this corollary are immediately established from the computations done explicitly in the proof of Proposition 5.3. We compute,

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{R}}(E_2), L_1]|_q \\
=& -\mathscr{L}_{\mathrm{R}}(\nabla_{E_1} E_2)|_q + \mathscr{L}_{\mathrm{NS}}(-\Gamma^2_{(3,1)} E_3)|_q + \nu(AR(E_2\wedge E_1) - \hat{R}(AE_2\wedge 0)A) \\
&+ \mathscr{L}_{\mathrm{NS}}(A(\star(\Gamma^1_{(2,3)} E_1 + \Gamma^1_{(1,2)} E_3))E_2)|_q \\
&- \nu\big(A\star(\Gamma^1_{(2,3)}(-\Gamma^2_{(3,1)} E_3) + \Gamma^1_{(1,2)}(\Gamma^2_{(3,1)} E_1))\big)\big|_q \\
&- \nu\big(A\star(E_2(\Gamma^1_{(2,3)})E_1 + E_2(\Gamma^1_{(1,2)})E_3)\big)|_q \\
=& -\mathscr{L}_{\mathrm{R}}(\nabla_{E_1} E_2)|_q - \Gamma^2_{(3,1)} L_3|_q + K\nu(A\star E_3)|_q + \mathscr{L}_{\mathrm{NS}}(A(\Gamma^1_{(2,3)} E_3 - \Gamma^1_{(1,2)} E_1))|_q \\
&- \nu\big(A\star(E_2(\Gamma^1_{(2,3)})E_1 + E_2(\Gamma^1_{(1,2)})E_3)\big)|_q \\
=& -\mathscr{L}_{\mathrm{R}}(\nabla_{E_1} E_2)|_q - \Gamma^2_{(3,1)} L_3|_q + \mathscr{L}_{\mathrm{R}}(\Gamma^1_{(2,3)} E_3 - \Gamma^1_{(1,2)} E_1)|_q - \Gamma^1_{(2,3)} L_3 + \Gamma^1_{(1,2)} L_1 \\
&+ (2\Gamma^1_{(2,3)}\Gamma^1_{(1,2)} - E_2(\Gamma^1_{(2,3)}))\nu(A\star E_1)|_q \\
&+ (-E_2(\Gamma^1_{(1,2)}) + K - (\Gamma^1_{(2,3)})^2 + (\Gamma^1_{(1,2)})^2)\nu(A\star E_3)|_q.
\end{aligned}$$

One knows from Eq. (57) that $-K = -E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2 - (\Gamma^1_{(2,3)})^2$ and $-E_2(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)} = 0$ and since also $\nabla_{E_1} E_2 = -\Gamma^1_{(1,2)} E_1 + \Gamma^1_{(2,3)} E_3$, this simplifies to

$$[\mathscr{L}_{\mathrm{R}}(E_2), L_1]_q = -\Gamma^2_{(3,1)} L_3|_q - \Gamma^1_{(2,3)} L_3 + \Gamma^1_{(1,2)} L_1.$$

The Lie bracket $[\mathscr{L}_{\mathrm{R}}(E_2), L_3]_q$ can be found by similar computations. We compute

$[\mathscr{L}_{\mathrm{R}}(E_3), L_1]|_q$. We have, recalling that $E_i(\Gamma^1_{(2,3)}) = 0$, $E_i(\Gamma^1_{(2,3)}) = 0$ for $i = 1, 3$,

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{R}}(E_3), L_1]|_q \\
=& -\mathscr{L}_{\mathrm{R}}(\nabla_{E_1} E_3)|_q + \mathscr{L}_{\mathrm{NS}}(\Gamma^1_{(2,3)} E_2 - \Gamma^3_{(3,1)} E_3)|_q \\
&+ \nu(AR(E_3 \wedge E_1)|_q - \hat{R}(AE_3 \wedge 0)|_q \\
&+ \mathscr{L}_{\mathrm{NS}}(A(\star(\Gamma^1_{(2,3)} E_1 + \Gamma^1_{(1,2)} E_3))E_3)|_q \\
&- \nu\big(A \star (\Gamma^1_{(2,3)}(\Gamma^1_{(2,3)} E_2 - \Gamma^3_{(3,1)} E_3) + \Gamma^1_{(1,2)}(\Gamma^3_{(3,1)} E_1 + \Gamma^1_{(1,2)} E_2))\big|_q \\
=& -\mathscr{L}_{\mathrm{R}}(\nabla_{E_1} E_3)|_q + (-K_2 - (\Gamma^1_{(2,3)})^2 - (\Gamma^1_{(1,2)})^2)\nu(A \star E_2)|_q \\
&- \Gamma^3_{(3,1)} L_3|_q - \Gamma^1_{(2,3)} \mathscr{L}_{\mathrm{R}}(E_2)|_q + 2L_2|_q.
\end{aligned}$$

The computation of $[\mathscr{L}_{\mathrm{R}}(E_1), L_3]|_q$ is similar. We compute $[L_3, L_1]$ with the following 4 steps:

$$\begin{aligned}
[\mathscr{L}_{\mathrm{NS}}(E_3), \mathscr{L}_{\mathrm{NS}}(E_1)]|_q =& \mathscr{L}_{\mathrm{NS}}(-\Gamma^1_{(3,1)} E_1 + 2\Gamma^1_{(2,3)} E_2 - \Gamma^3_{(3,1)} E_3)|_q \\
&+ \nu(AR(E_3 \wedge E_1) - \hat{R}(0 \wedge 0)A)|_q,
\end{aligned}$$

$$\begin{aligned}
&\big[\mathscr{L}_{\mathrm{NS}}(E_3), \nu\big((\cdot) \star (\Gamma^1_{(2,3)} E_1 + \Gamma^1_{(1,2)} E_3)\big)\big]|_q \\
=& \nu\big(A \star (\Gamma^1_{(2,3)}(\Gamma^1_{(2,3)} E_2 - \Gamma^3_{(3,1)} E_3) + \Gamma^1_{(1,2)}(\Gamma^3_{(3,1)} E_1 + \Gamma^1_{(1,2)} E_2))\big)\big|_q,
\end{aligned}$$

$$\begin{aligned}
&\big[\nu\big((\cdot) \star (-\Gamma^1_{(1,2)} E_1 + \Gamma^1_{(2,3)} E_3)\big), \mathscr{L}_{\mathrm{NS}}(E_1)\big]|_q \\
=& -\nu\big(A \star (-\Gamma^1_{(1,2)}(\Gamma^1_{(1,2)} E_2 - \Gamma^1_{(3,1)} E_3) + \Gamma^1_{(2,3)}(\Gamma^1_{(3,1)} E_1 - \Gamma^1_{(2,3)} E_2))\big)\big|_q,
\end{aligned}$$

$$\begin{aligned}
&\big[\nu\big((\cdot) \star (-\Gamma^1_{(1,2)} E_1 + \Gamma^1_{(2,3)} E_3)\big), \nu\big((\cdot) \star (\Gamma^1_{(2,3)} E_1 + \Gamma^1_{(1,2)} E_3)\big)\big]|_q \\
=& \nu\big(A\big[\star(-\Gamma^1_{(1,2)} E_1 + \Gamma^1_{(2,3)} E_3), \star(\Gamma^1_{(2,3)} E_1 + \Gamma^1_{(1,2)} E_3)\big]_{\mathfrak{so}}\big)\big|_q \\
=& ((\Gamma^1_{(1,2)})^2 + (\Gamma^1_{(2,3)})^2)\nu(A \star E_2)|_q.
\end{aligned}$$

Collecting these gives,

$$\begin{aligned}
[L_3, L_1]|_q =& -\Gamma^1_{(3,1)} L_1|_q - \Gamma^3_{(3,1)} L_3|_q + 2\Gamma^1_{(2,3)} \mathscr{L}_{\mathrm{NS}}(E_2)|_q \\
&- (K_2 + (\Gamma^1_{(2,3)})^2 + (\Gamma^1_{(1,2)})^2)\nu(A \star E_2)|_q.
\end{aligned}$$

□

5.2 Proof of Theorem 5.1

In this subsection, we prove Theorem 5.1. We therefore fix for the rest of the paragraph a non open orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, for some $q_0 \in Q$. By Proposition 4.2, one has that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) < 9 = \dim Q$ and, by Corollary 4.14, one knows that the rank of Rol_q is less than or equal to two, for every $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$.

For $j = 0, 1, 2$, let O_j be the set of points of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ where rank Rol_q is locally equal to j, i.e.,

$$O_j = \{q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \mid \text{there exists an open neighbourhood } O \text{ of } q \text{ in } \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \text{ such that rank } \mathsf{Rol}_{q'} = j,\ \forall q' \in O\}.$$

Notice that the union of the O_j's, when $j = 0, 1, 2$, is an open and dense subset of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ since each O_j is open in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ (but might be empty). Clearly, Item (a) in Theorem 5.1 describes the local structures of (M, g) and $(\hat{M}, \hat{g})$ at a point $q \in O_0$. The rest of the argument consists in addressing the same issue, first for $q \in O_2$ and then $q \in O_1$.

5.2.1 Local Structures for the Manifolds Around $q \in O_2$

Throughout the subsection, we assume, if not otherwise stated, that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open in Q (i.e., $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) < 9 = \dim Q$) and, in the statements involving O_2, the latter is non empty. Note that O_2 is also equal to the set of points of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ where rank Rol_q is equal to 2.

Proposition 5.6 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ so that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open in Q. Then, for every $q = (x, \hat{x}; A) \in O_2$, there exist an orthonormal pair $X_A, Y_A \in T|_x M$ such that if $Z_A := \star(X_A \wedge Y_A)$ then X_A, Y_A, Z_A is a positively oriented orthonormal pair with respect to which R and $\widetilde{\mathsf{Rol}}$ are written as

$$R(X_A \wedge Y_A) = \begin{pmatrix} 0 & K(x) & 0 \\ -K(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \star R(X_A \wedge Y_A) = \begin{pmatrix} 0 \\ 0 \\ -K(x) \end{pmatrix},$$
$$R(Y_A \wedge Z_A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K_1(x) \\ 0 & -K_1(x) & 0 \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_1(x) \\ 0 \\ 0 \end{pmatrix},$$
$$R(Z_A \wedge X_A) = \begin{pmatrix} 0 & 0 & -K_2(x) \\ 0 & 0 & 0 \\ K_2(x) & 0 & 0 \end{pmatrix}, \quad \star R(Z_A \wedge X_A) = \begin{pmatrix} 0 \\ -K_2(x) \\ 0 \end{pmatrix},$$
$$\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A) = 0,$$
$$\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) = \begin{pmatrix} 0 & 0 & -\alpha(q) \\ 0 & 0 & K_1^{\mathsf{Rol}}(q) \\ \alpha(q) & -K_1^{\mathsf{Rol}}(q) & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) = \begin{pmatrix} -K_1^{\mathsf{Rol}}(q) \\ -\alpha(q) \\ 0 \end{pmatrix},$$
$$\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} 0 & 0 & -K_2^{\mathsf{Rol}}(q) \\ 0 & 0 & \alpha(q) \\ K_2^{\mathsf{Rol}}(q) & -\alpha(q) & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} -\alpha(q) \\ -K_2^{\mathsf{Rol}}(q) \\ 0 \end{pmatrix},$$

where K, K_1, K_2 are real valued functions defined on M.

Consequently, with respect to the orthonormal oriented basis X_A, Y_A, Z_A of $T|_{\hat{x}}\hat{M}$

$$\star A^{\overline{T}} \hat{R}(AX_A \wedge AY_A)A = \begin{pmatrix} 0 \\ 0 \\ -K(x) \end{pmatrix},$$
$$\star A^{\overline{T}} \hat{R}(AY_A \wedge AZ_A)A = \begin{pmatrix} -K_1(x) + K_1^{\mathsf{Rol}}(q) \\ \alpha(q) \\ 0 \end{pmatrix},$$
$$\star A^{\overline{T}} \hat{R}(AZ_A \wedge AX_A)A = \begin{pmatrix} \alpha(q) \\ -K_2(x) + K_2^{\mathsf{Rol}}(q) \\ 0 \end{pmatrix}. \tag{38}$$

Before pursuing to the proof, we fix some additional notations provided in the following remark.

Remark 5.7 By the last proposition, $-K_1(x), -K_2(x), -K(x)$ are the eigenvalues of $R|_x$ corresponding to eigenvectors $\star X_A, \star Y_A, \star Z_A$ given by Proposition 5.6, for $q =$

$(x,\hat{x};A)\in O_2$. Recall that $Q(M,\hat{M})\to Q(\hat{M},M)$, $q=(x,\hat{x};A)\mapsto\hat{q}=(\hat{x},x;A^{\overline{T}})$ is an diffeomorphism which maps $\mathcal{D}_{\mathrm{R}}$ to $\widehat{\mathcal{D}_{\mathrm{R}}}$, where the latter is the rolling distribution on $Q(\hat{M},M)$. Hence this map maps $\mathcal{D}_{\mathrm{R}}$-orbits $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ to $\widehat{\mathcal{D}_{\mathrm{R}}}$-orbits $\mathcal{O}_{\widehat{\mathcal{D}_{\mathrm{R}}}}(\hat{q})$, for all $q\in Q$. So the rolling problem (R) is completely symmetric w.r.t. the changing of the roles of (M,g) and $(\hat{M},\hat{g})$. Hence Proposition 5.6 gives, when the roles of (M,g), $(\hat{M},\hat{g})$ are changed, for every $q=(x,\hat{x};A)\in O_2$ vectors $\hat{X}_A,\hat{Y}_A,\hat{Z}_A\in T|_{\hat{x}}\hat{M}$ such that $\widetilde{\mathsf{Rol}}_q((A^{\overline{T}}X_A)\wedge(A^{\overline{T}}\hat{Y}_A))=0$ and that $\hat{\star}\hat{X}_A,\hat{\star}\hat{Z}_A,\hat{\star}\hat{Z}_A$ are eigenbasis of $\hat{R}|_{\hat{x}}$ with eigenvalues which we call $-\hat{K}_1(\hat{x}),-\hat{K}_2(\hat{x}),-\hat{K}(\hat{x})$, respectively. The condition $\widetilde{\mathsf{Rol}}_q(X_A\wedge Y_A)=0$ implies that $K(x)=\hat{K}(\hat{x})$ for every $q=(x,\hat{x};A)\in O_2$ and also that $AZ_A=\hat{Z}_A$, since $\star(X_A\wedge Y_A)=Z_A$, $\hat{\star}(\hat{X}_A\wedge\hat{Y}_A)=\hat{Z}_A$.

We divide the proof of Proposition 5.6 into several lemmas.

Lemma 5.8 For every $q=(x,\hat{x};A)\in O_2$ and any orthonormal pair (which exists) $X_A,Y_A\in T|_xM$ such that $\mathsf{Rol}(X_A\wedge Y_A)=0$ and $X_A,Y_A,Z_A:=\star(X_A\wedge Y_A)$ is an oriented orthonormal basis of $T|_xM$, one has with respect to the basis X_A,Y_A,Z_A,

$$
\begin{aligned}
R(X_A\wedge Y_A)&=\begin{pmatrix}0 & K_A & \eta_A\\ -K_A & 0 & -\beta_A\\ -\eta_A & \beta_A & 0\end{pmatrix}, \quad \star R(X_A\wedge Y_A)=\begin{pmatrix}\beta_A\\ \eta_A\\ -K_A\end{pmatrix},\\
R(Y_A\wedge Z_A)&=\begin{pmatrix}0 & -\beta_A & \xi_A\\ \beta_A & 0 & K_A^1\\ -\xi_A & -K_A^1 & 0\end{pmatrix}, \quad \star R(Y_A\wedge Z_A)=\begin{pmatrix}-K_A^1\\ \xi_A\\ \beta_A\end{pmatrix},\\
R(Z_A\wedge X_A)&=\begin{pmatrix}0 & -\eta_A & -K_A^2\\ \eta_A & 0 & -\xi_A\\ K_A^2 & \xi_A & 0\end{pmatrix}, \quad \star R(Z_A\wedge X_A)=\begin{pmatrix}\xi_A\\ -K_A^2\\ \eta_A\end{pmatrix},\\
\widetilde{\mathsf{Rol}}_q&(X_A\wedge Y_A)=0,\\
\widetilde{\mathsf{Rol}}_q(Y_A\wedge Z_A)&=\begin{pmatrix}0 & 0 & -\alpha\\ 0 & 0 & K_1^{\mathsf{Rol}}\\ \alpha & -K_1^{\mathsf{Rol}} & 0\end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Y_A\wedge Z_A)=\begin{pmatrix}-K_1^{\mathsf{Rol}}\\ -\alpha\\ 0\end{pmatrix},\\
\widetilde{\mathsf{Rol}}_q(Z_A\wedge X_A)&=\begin{pmatrix}0 & 0 & -K_2^{\mathsf{Rol}}\\ 0 & 0 & \alpha\\ K_2^{\mathsf{Rol}} & -\alpha & 0\end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Z_A\wedge X_A)=\begin{pmatrix}-\alpha\\ -K_2^{\mathsf{Rol}}\\ 0\end{pmatrix}.
\end{aligned}
$$

Here $\eta_A,\beta_A,\xi_A,\alpha,K_1^{\mathsf{Rol}},K_2^{\mathsf{Rol}}$ depend *a priori* on the basis X_A,Y_A,Z_A and on the point q but the choice of these functions can be made *locally smoothly* on O_2 i.e., every $q\in O_2$ admits an open neighbourhood O_2' in O_2 such that the selection of these functions can be performed smoothly on O_2'.

Proof. Since rank $\mathsf{Rol}_q=2<3$ for $q\in O_2$, it follows that there is a unit vector $\omega_A\in\wedge^2T|_xM$ such that $\mathsf{Rol}_q(\omega_A)=0$. But in dimension 3, as mentioned in Appendix, one then has an orthonormal pair $X_A,Y_A\in T|_xM$ such that $\omega_A=X_A\wedge Y_A$. Moreover, the assignments $q\mapsto\omega_A,X_A,Y_A$ can be made locally smoothly. Set $Z_A:=\star(X_A\wedge Y_A)$. the fact that $\widetilde{\mathsf{Rol}}_q$ is a symmetric map implies that

$$
\begin{aligned}
g(\widetilde{\mathsf{Rol}}_q(Y_A\wedge Z_A),X_A\wedge Y_A)=&g(\widetilde{\mathsf{Rol}}_q(X_A\wedge Y_A),Y_A\wedge Z_A)=0,\\
g(\widetilde{\mathsf{Rol}}_q(Z_A\wedge X_A),X_A\wedge Y_A)=&g(\widetilde{\mathsf{Rol}}_q(X_A\wedge Y_A),Z_A\wedge X_A)=0.
\end{aligned}
$$

□

As a consequence of the previous result and because, for $X, Y \in T|_x M$, one gets

$$A^{\overline{T}} \hat{R}(AX \wedge AY)A = R(X \wedge Y) - \widetilde{\mathsf{Rol}}_q(X \wedge Y),$$

then we have that, w.r.t. the oriented orthonormal basis AX_A, AY_Z, AZ_A of $T|_{\hat{x}}\hat{M}$,

$$\begin{aligned}
\hat{\star} A^{\overline{T}} \hat{R}(AX_A \wedge AY_A)A &= \begin{pmatrix} \beta_A \\ \eta_A \\ -K_A \end{pmatrix}, \\
\hat{\star} A^{\overline{T}} \hat{R}(AY_A \wedge AZ_A)A &= \begin{pmatrix} -K_A^1 + K_1^{\mathsf{Rol}} \\ \xi_A + \alpha \\ \beta_A \end{pmatrix}, \\
\hat{\star} A^{\overline{T}} \hat{R}(AZ_A \wedge AX_A)A &= \begin{pmatrix} \xi_A + \alpha \\ -K_A^2 + K_2^{\mathsf{Rol}} \\ \eta_A \end{pmatrix}.
\end{aligned} \tag{39}$$

The assumption that rank $\mathsf{Rol}_q = 2$ on O_2 is equivalent to the fact that for any choice of X_A, Y_A, Z_A as above, $\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A)$ and $\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A)$ are linearly independent for every $q = (x, \hat{x}; A) \in O_2$ i.e.

$$K_1^{\mathsf{Rol}}(q) K_2^{\mathsf{Rol}}(q) - \alpha(q)^2 \neq 0. \tag{40}$$

We next show that, with any (non-unique) choice of X_A, Y_A as in Lemma 5.8, then $\eta_A = \beta_A = 0$.

Lemma 5.9 Choose any $X_A, Y_A, Z_A = \star(X_A \wedge Y_A)$ as in Lemma 5.8. Then, for every $q = (x, \hat{x}; A) \in O_2$ and any vector fields $X, Y, Z, W \in \mathrm{VF}(M)$, one has

$$\big[\nu(\mathsf{Rol}(X \wedge Y)(\cdot)), \nu(\mathsf{Rol}(Z \wedge W)(\cdot))\big]\big|_q \in \nu(\mathrm{span}\{\star X_A, \star Y_A\})|_q \subset T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0). \tag{41}$$

Moreover, $\pi_Q|_{O_2}$ is an submersion (onto an open subset of $M \times \hat{M}$), $\dim V|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) = 2$ for all $q \in O_2$ and $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

Proof. First notice that by Lemma 5.8

$$\begin{pmatrix} \mathsf{Rol}_q(\star X_A) \\ \mathsf{Rol}_q(\star Y_A) \end{pmatrix} = \begin{pmatrix} -K_1^{\mathsf{Rol}} & -\alpha \\ -\alpha & -K_2^{\mathsf{Rol}} \end{pmatrix} \begin{pmatrix} \star X_A \\ \star Y_A \end{pmatrix}$$

for $q = (x, \hat{x}; A) \in O_2$ and since the determinant of the matrix on the right hand side is, at $q \in O_2$, $K_1^{\mathsf{Rol}}(q) K_2^{\mathsf{Rol}}(q) - \alpha(q)^2 \neq 0$, as noticed in (40) above, it follows that

$$\star X_A, \star Y_A \in \mathrm{span}\{\mathsf{Rol}_q(\star X_A), \mathsf{Rol}_q(\star Y_A)\}.$$

Next, from Proposition 4.8 we know that, for every $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and every $Z, W \in T|_x M$

$$\nu(\mathsf{Rol}_q(Z \wedge W))|_q \in V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}) \subset T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

Hence, $\nu(\mathsf{Rol}_q(\star X_A)),\nu(\mathsf{Rol}_q(\star Y_A))\in V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})$ for every $q\in O_2$ and then

$$\nu(A\star X_A),\nu(A\star Y_A)\in V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}), \tag{42}$$

for all $q=(x,\hat{x};A)\in O_2$. We claim that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}|_{O_2}$ is a submersion (onto an open subset of $M\times\hat{M}$). Indeed, for any vector field $W\in\mathrm{VF}(M)$ one has $\mathscr{L}_{\mathrm{R}}(W)|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for $q=(x,\hat{x};A)\in O_2$ and since the assignments $q\mapsto X_A,Y_A$ can be made locally smoothly, then also $[\mathscr{L}_{\mathrm{R}}(W),\nu(A\star X_A)]|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. But then Proposition 3.36 implies that

$$\begin{aligned}
&(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*([\mathscr{L}_{\mathrm{R}}(W),\nu(A\star X_A)]|_q)\\
=&(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*\big(-\mathscr{L}_{\mathrm{NS}}(A(\star X_A)W)|_q+\nu(A\star\mathscr{L}_{\mathrm{R}}(W)|_qX_{(\cdot)})|_q\big)\\
=&(0,-A(\star X_A)W),
\end{aligned}$$

where we wrote $X_{(\cdot)}$ as for the map $q\mapsto X_A$. Similarly,

$$(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*([\mathscr{L}_{\mathrm{R}}(W),\nu(A\star Y_A)]|_q)=(0,-A(\star Y_A)W).$$

This shows that for all $q=(x,\hat{x};A)\in O_2$ and $Z,W\in T|_xM$, we have

$$(0,-A(\star X_A)W),(0,-A(\star Y_A)W)\in(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset T|_xM\times T|_{\hat{x}}\hat{M}.$$

Because $\star X_A,\star Y_A$ are linearly independent, this implies that

$$\{0\}\times T|_{\hat{x}}\hat{M}\subset(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

Finally, because $\mathscr{L}_{\mathrm{R}}(W)|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for any $q=(x,\hat{x};A)\in\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and any $W\in T|_xM$, and $(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*\mathscr{L}_{\mathrm{R}}(W)|_q=(W,AW)$, one also has

$$(W,0)=(W,AW)-(0,AW)\in(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),$$

which implies

$$T|_xM\times\{0\}\subset(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})_*T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

This proves that $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}|_{O_2}=\pi_Q|_{O_2}$ is indeed a submersion.

Because O_2 is not open in Q (otherwise $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ would be an open subset of Q), it follows that $\dim O_2\leq 8$ and since $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}|_{O_2}$ has rank 6, being a submersion, we deduce that for all $q\in O_2$,

$$\dim V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})=\dim O_2-6\leq 2.$$

But because of (42) we see that $\dim V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})\geq 2$ i.e.

$$\dim V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)})=2,$$

which shows that $\dim O_2=8$, hence $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=8$ and

$$\mathrm{span}\{\nu(A\star X_A)|_q,\nu(A\star Y_A)|_q\}=V|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)),\quad\forall q=(x,\hat{x};A)\in O_2.$$

To conclude the proof, it is enough to notice that since for any $X,Y,Z,W\in\mathrm{VF}(M)$, $\nu(\mathsf{Rol}(X\wedge Y)(A))|_q,\nu(\mathsf{Rol}(Z\wedge W)(A))|_q\in V|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, then

$$[\nu(\mathsf{Rol}(X\wedge Y)(\cdot)),\nu(\mathsf{Rol}(Z\wedge W)(\cdot))]|_q\in V|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)).$$

□

Lemma 5.10 If one chooses any $X_A, Y_A, Z_A = \star(X_A \wedge Y_A)$ as in Lemma 5.8, then

$$\eta_A = \beta_A = 0, \quad \forall q = (x, \hat{x}; A) \in O_2.$$

Proof. Fix $q = (x, \hat{x}; A) \in O_2$. Choosing in Corollary 4.14 $X, Y \in \mathrm{VF}(M)$ such that $X|_x = X_A$, $Y|_x = Y_A$, we get, since $\mathsf{Rol}_q(X_A \wedge Y_A) = 0$,

$$\begin{aligned}
&\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X \wedge Y)(\cdot)), \nu(\mathsf{Rol}(Z \wedge W)(\cdot))\big]\big|_q \\
=&A\big[R(X_A \wedge Y_A), R(Z|_x \wedge W|_x)\big]_{\mathfrak{so}} - \big[\hat{R}(AX_A \wedge AY_A), \hat{R}(AZ|_x \wedge AW|_x)\big]_{\mathfrak{so}} A \\
&+ \hat{R}(AX_A, A\widetilde{\mathsf{Rol}}_q(Z|_x \wedge W|_x)Y_A)A + \hat{R}(A\widetilde{\mathsf{Rol}}_q(Z|_x \wedge W|_x)X_A, AY_A)A.
\end{aligned}$$

We compute the right hand side of this formula in in two special cases (a)-(b) below.

(a) Take $Z, W \in \mathrm{VF}(M)$ such that $Z|_x = Y_A$, $W|_x = Z_A$.

In this case, computing the matrices in the basis $\star X_A, \star Y_A, \star Z_A$,

$$\begin{aligned}
&A^{\overline{T}}\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X \wedge Y)(\cdot)), \nu(\mathsf{Rol}(Z \wedge W)(\cdot))\big]\big|_q \\
=&\big[R(X_A \wedge Y_A), R(Y_A \wedge Z_A)\big]_{\mathfrak{so}} - A^{\overline{T}}\big[\hat{R}(AX_A \wedge AY_A), \hat{R}(AY_A \wedge AZ_A)\big]_{\mathfrak{so}} A \\
&+ A^{\overline{T}}\hat{R}(AX_A, A\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A)Y_A)A + A^{\overline{T}}\hat{R}(A\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A)X_A, AY_A)A \\
=&\begin{pmatrix} \beta_A \\ \eta_A \\ -K_A \end{pmatrix} \wedge \begin{pmatrix} -K_A^1 \\ \xi_A \\ \beta_A \end{pmatrix} - \begin{pmatrix} \beta_A \\ \eta_A \\ -K_A \end{pmatrix} \wedge \begin{pmatrix} -K_A^1 + K_1^{\mathsf{Rol}} \\ \xi_A + \alpha \\ \beta_A \end{pmatrix} \\
&+ A^{\overline{T}}\hat{R}(AX_A, -K_1^{\mathsf{Rol}}AZ_A)A + A^{\overline{T}}\hat{R}(\alpha AZ_A, AY_A)A \\
=&-\begin{pmatrix} \beta_A \\ \eta_A \\ -K_A \end{pmatrix} \wedge \begin{pmatrix} K_1^{\mathsf{Rol}} \\ \alpha \\ 0 \end{pmatrix} + K_1^{\mathsf{Rol}}\begin{pmatrix} \xi_A + \alpha \\ -K_A^2 + K_2^{\mathsf{Rol}} \\ \eta_A \end{pmatrix} - \alpha\begin{pmatrix} -K_A^1 + K_1^{\mathsf{Rol}} \\ \xi_A + \alpha \\ \beta_A \end{pmatrix} \\
=&\begin{pmatrix} -\alpha K_A + K_1^{\mathsf{Rol}}(\xi_A + \alpha) - \alpha(-K_A^1 + K_1^{\mathsf{Rol}}) \\ K_A K_1^{\mathsf{Rol}} + K_1^{\mathsf{Rol}}(-K_A^2 + K_2^{\mathsf{Rol}}) - \alpha(\xi_A + \alpha) \\ -\alpha\beta_A + K_1^{\mathsf{Rol}}\alpha_A + K_1^{\mathsf{Rol}}\alpha_A - \alpha\beta_A \end{pmatrix} = \begin{pmatrix} \star \\ \star \\ 2(K_1^{\mathsf{Rol}}\eta_A - \alpha\beta_A) \end{pmatrix}.
\end{aligned}$$

By Lemma 5.9 the right hand side should belong to the span of $\star X_A, \star Y_A$ which implies

$$K_1^{\mathsf{Rol}}\eta_A - \alpha\beta_A = 0. \tag{43}$$

(b) Take $Z, W \in \mathrm{VF}(M)$ such that $Z|_x = Z_A$, $W|_x = X_A$.

Again, computing w.r.t. the basis $\star X_A, \star Y_A, \star Z_A$, yields

$$
\begin{aligned}
&A^{\overline{T}}\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X,Y)(\cdot)),\nu(\mathsf{Rol}(Z,W)(\cdot))\big]\big|_q\\
=&\big[R(X_A,Y_A),R(Z_A,X_A)\big]_{\mathfrak{so}}-A^{\overline{T}}\big[\hat{R}(AX_A,AY_A),\hat{R}(AZ_A,AX_A)\big]_{\mathfrak{so}}A\\
&+A^{\overline{T}}\hat{R}(AX_A,A\widetilde{\mathsf{Rol}}_q(Z_A,X_A)Y_A)A+A^{\overline{T}}\hat{R}(A\widetilde{\mathsf{Rol}}_q(Z_A,X_A)X_A,AY_A)A\\
=&\Big(\begin{pmatrix}\beta_A\\ \alpha_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}\xi_A\\ -K_A^2\\ \eta_A\end{pmatrix}-\begin{pmatrix}\beta_A\\ \eta_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}\xi_A+\alpha\\ -K_A^2+K_2^{\mathsf{Rol}}\\ \eta_A\end{pmatrix}\Big)\\
&+A^{\overline{T}}\hat{R}(AX_A,-\alpha AZ_A)A+A^{\overline{T}}\hat{R}(K_2^{\mathsf{Rol}}AZ_A,AY_A)A\\
=&-\begin{pmatrix}\beta_A\\ \eta_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}\alpha\\ K_2^{\mathsf{Rol}}\\ 0\end{pmatrix}+\alpha\begin{pmatrix}\xi_A+\alpha\\ -K_A^2+K_2^{\mathsf{Rol}}\\ \eta_A\end{pmatrix}-K_2^{\mathsf{Rol}}\begin{pmatrix}-K_A^1+K_1^{\mathsf{Rol}}\\ \xi_A+\alpha\\ \beta_A\end{pmatrix}\\
=&\begin{pmatrix}-K_AK_2^{\mathsf{Rol}}+\alpha(\xi_A+\alpha)-K_2^{\mathsf{Rol}}(-K_A^1+K_1^{\mathsf{Rol}})\\ \alpha K_A+\alpha(-K_A^2+K_2^{\mathsf{Rol}})-K_2^{\mathsf{Rol}}(\xi_A+\alpha)\\ -\beta_AK_2^{\mathsf{Rol}}+\alpha\eta_A+\alpha\eta_A-K_2^{\mathsf{Rol}}\beta_A\end{pmatrix}=\begin{pmatrix}\star\\ \star\\ 2(\alpha\eta_A-\beta_AK_2^{\mathsf{Rol}})\end{pmatrix}.
\end{aligned}
$$

Since the right hand side belongs to the span of $\star X_A, \star Y_A$, by Lemma 5.9, we obtain

$$
\alpha\eta_A-K_2^{\mathsf{Rol}}\beta_A=0. \tag{44}
$$

Combining Equations (43) and (44) we get

$$
\begin{pmatrix}K_1^{\mathsf{Rol}} & \alpha\\ \alpha & K_2^{\mathsf{Rol}}\end{pmatrix}\begin{pmatrix}\eta_A\\ \beta_A\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.
$$

According to Eq. (40) the determinant of the 2×2-matrix on the left hand side does not vanish, which implies that $\eta_A=\beta_A=0$. The proof is finished. ☐

Lemma 5.11 For every $q=(x,\hat{x};A)\in O_2$, there are orthonormal $X_A, Y_A\in T|_xM$ such that $X_A, Y_A, Z_A=\star(X_A\wedge Y_A)$ is an oriented orthonormal basis of $T|_xM$ with respect to which in Lemma 5.8 one has

$$
\eta_A=\beta_A=\xi_A=0,
$$

i.e., $\star X_A, \star Y_A, \star Z_A$ are eigenvectors of $R|_x$.

Proof. Fix $q=(x,\hat{x};A)\in O_2$, choose any $X_A, Y_A, Z_A=\star(X_A\wedge Y_A)$ as in Lemma 5.8 and suppose $\xi_A\neq 0$ (otherwise we are done). By Lemma 5.10, one has $\eta_A=\beta_A=0$, meaning that $\star Z_A$ is an eigenvector of $R|_x$. For $t\in\mathbb{R}$, set

$$
\begin{pmatrix}X_A(t)\\ Y_A(t)\end{pmatrix}:=\begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}X_A\\ Y_A\end{pmatrix}.
$$

Then clearly $Z_A(t):=\star(X_A(t)\wedge Y_A(t))=\star(X_A\wedge Y_A)=Z_A$, and $X_A(t), Y_A(t), Z_A(t)$ is an orthonormal positively oriented basis of $T|_xM$. Since

$$
\mathsf{Rol}_q(\star Z_A(t))=\mathsf{Rol}_q(\star Z_A)=0,
$$

Lemma 5.10 implies that $\eta_A(t),\beta_A(t)=0$ if one writes $\eta_A(t),\beta_A(t),\xi_A(t)$ for the coefficients of matrices in Lemma 5.8 w.r.t $X_A(t), Y_A(t), Z_A(t)$. Our goal is to show that $\xi_A(t)=0$ for some $t\in\mathbb{R}$.

First of all $\star Z_A(t) = \star Z_A$ is a unit eigenvector of $R|_x$ which does not depend on t. On the other hand, $R|_x$ is a symmetric map $\wedge^2 T|_x M \to \wedge^2 T|_x M$, so it has two orthogonal unit eigenvectors, say, u_1, u_2 in $(\star Z_A)^\perp = \star(Z_A^\perp)$. Thus $u_1, u_2, \star Z_A$ forms an orthonormal basis of $\wedge^2 T|_x M$, which we may assume to be oriented (otherwise swap u_1, u_2). Then $\mathrm{span}\{u_1, u_2\} = \star Z_A^\perp = \mathrm{span}\{\star X_A, \star Y_A\}$ and there exists $t_0 \in \mathbb{R}$ such that $\star X_A(t_0) = u_1$, $\star Y_A(t_0) = u_2$ Since $R|_x(\star X_A(t_0)) = -K_1 \star X_A(t_0)$, $R|_x(\star Y_A(t_0)) = -K_2 \star Y_A(t_0)$, we have $\xi_A(t_0) = 0$ as well as $\eta_A(t_0) = \beta_A(t_0) = 0$. $\square$

Remark 5.12 Notice that the choice of Z_A can be made locally smoothly on O_2 but, at this stage of the argument, it is not clear that one can choose X_A, Y_A, with $\xi_A = 0$, locally smoothly on O_2. However, it will be the case cf. Corollary 5.16.

We now aim to prove, roughly speaking, that the eigenvalue $-K$ must be double for both spaces (M, g), $(\hat{M}, \hat{g})$ if neither one of them has constant curvature.

Lemma 5.13 If the eigenspace at $x_1 \in \pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O_2)$ corresponding to the eigenvalue $-K(x_1)$ of the curvature operator R is of dimension one, then $(\hat{M}, \hat{g})$ has constant curvature $K(x_1)$ on the open set $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}}(\pi^{-1}_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(x_1))$ of $\hat{M}$. The claim also holds with the roles of (M, g) and $(\hat{M}, \hat{g})$ interchanged.

Proof. Suppose that at $x_1 \in \pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O_2)$ the eigenvalue $-K(x_1)$ has multiplicity 1. By continuity, the $-K(\cdot)$-eigenspace of R is of dimension one on an open neighbourhood U of x_1. Since this eigenspace depends smoothly on a point of M, we may choose, taking U smaller around x_1 if needed, positively oriented orthonormal smooth vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on U such that $\star\tilde{Z} = \tilde{X} \wedge \tilde{Y}$ spans the $-K(\cdot)$-eigenspace of R at each point of U. Taking arbitrary $q' = (x', \hat{x}'; A') \in (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U) \cap O_2$ and letting $X_{A'}, Y_{A'}, Z_{A'}$ be the vectors provided by Theorem 5.8 at q, we have that the $-K(x')$-eigenspace of $R|_{x'}$ is also spanned by $X_{A'} \wedge Y_{A'}$. By the orthonormality and orientability, $X_{A'} \wedge Y_{A'} = \tilde{X}|_{x'} \wedge \tilde{Y}|_{x'}$ from which $\tilde{Z}|_{x'} = Z_{A'}$ and $\mathsf{Rol}(\tilde{X}|_{x'} \wedge \tilde{Y}|_{x'})(A') = \mathsf{Rol}(X_{A'} \wedge Y_{A'})(A') = 0$. Now fix, for a moment, $q = (x, \hat{x}; A) \in (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U) \cap O_2$. By replacing $\tilde{X}$ by $\cos(t)\tilde{X} + \sin(t)\tilde{Y}$ and $\tilde{Y}$ by $-\sin(t)\tilde{X} + \cos(t)\tilde{Y}$ on U for a certain constant $t = t_x \in \mathbb{R}$, we may assume that $\tilde{X}|_x = X_A$, $\tilde{Y}|_x = Y_A$. Since, as we just proved, for all $(x', \hat{x}'; A') \in (\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U) \cap O_2$, one has

$$\mathsf{Rol}(\tilde{X}|_{x'} \wedge \tilde{Y}|_{x'})(A') = 0,$$

then the vector field $\nu(\mathsf{Rol}(\tilde{X} \wedge \tilde{Y})(\cdot)) \in \mathrm{VF}(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})$ vanishes identically i.e. $\nu(\mathsf{Rol}(\tilde{X} \wedge \tilde{Y})(\cdot)) = 0$ on $(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U) \cap O_2$. Therefore, the computation in part (a) of the proof of Lemma 5.10 (replace $X \to \tilde{X}$, $Y \to \tilde{Y}$, $Z \to \tilde{Y}$, $W \to \tilde{Z}$ there; recall also that $\xi_A = 0$ by the choice of X_A, Y_A, Z_A) gives, by noticing also that here $K_A = K(x)$, $K_A^1 = K_1(x)$ and $K_A^2 = K_2(x)$,

$$\begin{aligned}
0 =& A^{\overline{T}} \nu|_q^{-1} \big[\nu(\mathsf{Rol}(\tilde{X}, \tilde{Y})(\cdot)), \nu(\mathsf{Rol}(\tilde{Y}, \tilde{Z})(\cdot))\big]\big|_q \\
=& \begin{pmatrix} -\alpha K_A + \alpha K_1^{\mathsf{Rol}} - \alpha(-K_A^1 + K_1^{\mathsf{Rol}}) \\ K_A K_1^{\mathsf{Rol}} + K_1^{\mathsf{Rol}}(-K_A^2 + K_2^{\mathsf{Rol}}) - \alpha^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha(-K + K_1) \\ K_1^{\mathsf{Rol}}(K - K_2 + K_2^{\mathsf{Rol}}) - \alpha^2 \\ 0 \end{pmatrix}.
\end{aligned}$$

Similarly, the computation in part (b) of the proof of Lemma 5.10 (now replace $X \to \tilde{X}$, $Y \to \tilde{Y}$, $Z \to \tilde{Z}$, $W \to \tilde{X}$ there) gives,

$$\begin{aligned} 0 =& A^{\overline{T}}\nu|_q^{-1}\big[\nu(\mathsf{Rol}(\tilde{X},\tilde{Y})(\cdot)),\nu(\mathsf{Rol}(\tilde{Z},\tilde{X})(\cdot))\big]\big|_q \\ =& \begin{pmatrix} -K_AK_2^{\mathsf{Rol}}+\alpha^2-K_2^{\mathsf{Rol}}(-K_A^1+K_1^{\mathsf{Rol}}) \\ \alpha K_A+\alpha(-K_A^2+K_2^{\mathsf{Rol}})-K_2^{\mathsf{Rol}}\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} K_2^{\mathsf{Rol}}(-K+K_1-K_1^{\mathsf{Rol}})+\alpha^2 \\ \alpha(K-K_2) \\ 0 \end{pmatrix}. \end{aligned}$$

By assumption, $-K(\cdot)$ is an eigenvalue of R distinct from the other eigenvalues $-K_1(\cdot)$, $-K_2(\cdot)$ on $U>$ Hence we must have $\alpha(q)=0$. Since $0 \neq K_1^{\mathsf{Rol}}(q)K_2^{\mathsf{Rol}}(q) - \alpha(q)^2 = K_1^{\mathsf{Rol}}(q)K_2^{\mathsf{Rol}}(q)$, we have $K_1^{\mathsf{Rol}}(q)\neq 0$ and $K_2^{\mathsf{Rol}}(q)\neq 0$, hence $K(x)-K_1(x)+K_1^{\mathsf{Rol}}(q)=0$ and $K(x)-K_2(x)+K_2^{\mathsf{Rol}}(q)=0$ for $q=(x,\hat{x};A)\in(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U)\cap O_2$. Since $q=(x,\hat{x};A)\in(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U)\cap O_2$ was arbitrary, we have proven that

$$\begin{aligned} \alpha(q)&=0, \\ -K_1(x)+K_1^{\mathsf{Rol}}(q)&=-K(x), \\ -K_2(x)+K_2^{\mathsf{Rol}}(q)&=-K(x), \end{aligned}$$

for all $q=(x,\hat{x};A)\in(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U)\cap O_2$.

Looking at (38) reveals that for every $q=(x,\hat{x};A)\in(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(U)\cap O_2$, the three 2-vectors $AX_A\wedge AY_A$, $AY_A\wedge AZ_A$ and $AZ_A\wedge AX_A$ are mutually orthonormal eigenvectors of $\hat{R}|_{\hat{x}}$ all corresponding to the eigenvalue $-K(x)$, i.e, $(\hat{M},\hat{g})$ has constant curvature $-K(x)$ at $\hat{x}$. Since $x_1\in U$, the Riemannian space $(\hat{M},\hat{g})$ has constant curvature $-K(x_1)$ at all points $\hat{x}_1\in\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}}\big((\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x_1)\cap O_2\big)$.

Finally, we argue that $\hat{S}:=\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\hat{M}}\big((\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x_1)\cap O_2\big)$ is an open subset of $\hat{M}$. It is enough to show that $\pi_{Q,\hat{M}}|_{\hat{O}_{x_1}}:\hat{O}_{x_1}\to\hat{M}$ is a submersion where $\hat{O}_{x_1}:=(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})^{-1}(x_1)\cap O_2$ is a submanifold of O_2. To begin with, recall that $\pi_Q|_{O_2}$ is an submersion onto an open subset of $M\times\hat{M}$ by Lemma 5.9. Let $q\in\hat{O}_{x_1}$ and write $q=(x_1,\hat{x};A)$. Choose any frame $\hat{X}_1,\hat{X}_2,\hat{X}_3$ of $T|_{\hat{x}}\hat{M}$. Then there are $\hat{W}_i\in T|_q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $i=1,2,3$, such that $(\pi_Q)_*(\hat{W}_i)=(0,\hat{X}_i)$. In particular, $(\pi_{Q,M})_*(\hat{W}_i)=0$, so $\hat{W}_i\in V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})$. But since $T|_q\hat{O}_{x_1}=V|_q(\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M})$, we have $\hat{W}_i\in T|_q\hat{O}_{x_1}$ and thus $\hat{X}_i=(\pi_{Q,\hat{M}})_*\hat{W}_i\in\mathrm{im}(\pi_{Q,\hat{M}}|_{\hat{O}_{x_1}})_*$, which proves the claim and finishes the proof. □

Remark 5.14 It is actually obvious that the eigenvalue $-K(\cdot)$ of R of (M,g) is constant, equal to $K(x_1)$ say, in a some neighbourhood of x_1 in M, if $-K(x_1)$ were a single eigenvalue of $R|_{x_1}$. Even more is true: One could show, even without questioning whether $-K(\cdot)$ is a single eigenvalue for R and/or $\hat{R}$ or not, that on $\pi_{Q,M}(O_2)$ and $\pi_{Q,\hat{M}}(O_2)$ this eigenvalue is actually locally constant (i.e. the function $K(\cdot)$ is locally constant). This fact will be observed e.g. in Lemma 5.17 below.

Lemma 5.15 The following holds:

(1) For any $q_1=(x_1,\hat{x}_1;A_1)\in O_2$, $(\hat{M},\hat{g})$ cannot have constant curvature at $\hat{x}_1$.

(2) There does not exist a $q_1=(x_1,\hat{x}_1;A_1)\in O_2$ such that $-K(x_1)$ is a single eigenvalue of $R|_{x_1}$.

This also holds with the roles of (M,g) and $(\hat{M},\hat{g})$ interchanged.

Proof. (1) Suppose $(\hat{M},\hat{g})$ has a constant curvature $\hat{K}$ at $\hat{x}_1$. Let E_1,E_2,E_3 be an oriented orthonormal frame on a neighbourhood U of x_1 such that $\star E_1|_{x_1},\star E_2|_{x_1},\star E_3|_{x_1}$ are eigenvectors of R at x_1 with eigenvalues $-K_1(x_1)$, $-K_2(x_1)$, $-K(x_1)$, respectively, where these eigenvalues are as in Proposition 5.6. As we have noticed, $\hat{K}=K(x_1)$. Because $\hat{R}|_{\hat{x}_1}=-\hat{K}\mathrm{id}_{\wedge^2 T|_{\hat{x}_1}\hat{M}}$, one has

$$\begin{aligned}
\widetilde{\mathsf{Rol}}_{q_1}(\star E_1) =&(-K_1(x_1)+\hat{K})\star E_1|_{x_1},\\
\widetilde{\mathsf{Rol}}_{q_1}(\star E_2) =&(-K_2(x_1)+\hat{K})\star E_2|_{x_1},\\
\widetilde{\mathsf{Rol}}_{q_1}(\star E_3) =&(-K(x_1)+\hat{K})\star E_3|_{x_1}=0.
\end{aligned}$$

Since rank $\widetilde{\mathsf{Rol}}_{q_1}=2$, we have $-K_1(x_1)+\hat{K}\neq 0$, $-K_2(x_1)+\hat{K}\neq 0$.

Because the vector fields $\nu(\mathsf{Rol}(\star E_1)(\cdot))$, $\nu(\mathsf{Rol}(\star E_2)(\cdot))$ are tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ on $O_2':=O_2\cap\pi_{Q,M}^{-1}(U)$, so is their Lie bracket. According to Proposition 3.37, the value of this bracket at q_1 is equal to

$$[\nu(\mathsf{Rol}(\star E_1)(\cdot)),\nu(\mathsf{Rol}(\star E_2)(\cdot))]|_{q_1}=(-K_1(x_1)+\hat{K})(-K_2(x_1)+\hat{K})\nu(A\star E_3)|_{q_1}.$$

Hence $\nu(\mathsf{Rol}(\star E_1)(\cdot)$, $\nu(\mathsf{Rol}(\star E_2)(\cdot)$, $[\nu(\mathsf{Rol}(\star E_1)(\cdot)),\nu(\mathsf{Rol}(\star E_2)(\cdot))]$ are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and since they are linearly independent at q_1, hence they are linearly independent on an open neighbourhood of q_1 in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Therefore, from Corollary 4.18, it follows that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is open in Q, which is a contradiction.

(2) Suppose $-K(x_1)$ is a single eigenvector of $R|_{x_1}$, where $q_1=(x_1,\hat{x}_1;A_1)\in O_2$. Then, by Lemma 5.13, the space $(\hat{M},\hat{g})$ would have a constant curvature in an open set which is a neighbourhood of $\hat{x}_1$. By Case (1), this leads to a contradiction. □

By the last two lemmas, we may thus assume that for every $q=(x,\hat{x};A)\in O_2$ the common eigenvalue $-K(x)=-\hat{K}(\hat{x})$ of $R|_x$, $\hat{R}|_{\hat{x}}$ has multiplicity two. It has the following consequence.

Corollary 5.16 The assignments $q\mapsto X_A,Y_A,Z_A$ and $q\mapsto K_1^{\mathsf{Rol}}(q),K_2^{\mathsf{Rol}}(q),\alpha(q)$ as in Proposition 5.6 can be made locally smoothly on O_2.

Proof. Let $q_1=(x_1,\hat{x}_1;A_1)\in O_2$. By Lemma 5.15, there are open neighbourhoods $U\ni x_1$ and $\hat{U}\ni\hat{x}_1$ such that the eigenvalues $-K_2(x)$ of $R|_x$ and $-\hat{K}_2(\hat{x})$ of $\hat{R}|_{\hat{x}}$ are both simple. Therefore the map $q\mapsto Y_A$ can be made locally smoothly on O_2 and this is also the case for the map $q\mapsto Z_A$ since it corresponds to the 1-dimensional kernel of $\widetilde{\mathsf{Rol}}_q$ and $X_A=\star(Y_A\wedge Z_A)$. □

Lemma 5.17 For every $q_1=(x_1,\hat{x}_1;A_1)\in O_2$, there are open neighbourhoods $U,\hat{U}$ of $x_1,\hat{x}_1$ and oriented orthonormal frames E_1,E_2,E_3 on M, $\hat{E}_1,\hat{E}_2,\hat{E}_3$ on $\hat{M}$ with respect to which the connections tables are of the form

$$\Gamma=\begin{pmatrix}\Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)}\\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)}\\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)}\end{pmatrix},\quad \hat{\Gamma}=\begin{pmatrix}\hat{\Gamma}^1_{(2,3)} & 0 & -\hat{\Gamma}^1_{(1,2)}\\ \hat{\Gamma}^1_{(3,1)} & \hat{\Gamma}^2_{(3,1)} & \hat{\Gamma}^3_{(3,1)}\\ \hat{\Gamma}^1_{(1,2)} & 0 & \hat{\Gamma}^1_{(2,3)}\end{pmatrix},$$

and

$$
\begin{aligned}
&V(\Gamma^1_{(2,3)})=0, \quad V(\Gamma^1_{(1,2)})=0, \quad \forall V\in E_2|_x^\perp, \quad x\in U,\\
&\hat{V}(\hat{\Gamma}^1_{(2,3)})=0, \quad \hat{V}(\hat{\Gamma}^1_{(1,2)})=0, \quad \forall \hat{V}\in \hat{E}_2|_{\hat{x}}^\perp, \quad \hat{x}\in \hat{U}.
\end{aligned}
$$

Moreover, $\star E_1,\star E_2,\star E_3$ are eigenvectors of R with eigenvalues $-K,-K_2(\cdot),-K$ on U and similarly $\hat{\star}\hat{E}_1,\hat{\star}\hat{E}_2,\hat{\star}\hat{E}_3$ are eigenvectors of $\hat{R}$ with eigenvalues $-K,-\hat{K}_2(\cdot),-K$ on $\hat{U}$, where $K\in\mathbb{R}$ is constant.

Proof. As we just noticed, for every $q=(x,\hat{x};A)\in O_2$, the common eigenvalue $-K(x)=-\hat{K}(\hat{x})$ of $R|_x$ and $\hat{R}|_{\hat{x}}$ has multiplicity equal to two.

Fix $q_1=(x_1,\hat{x}_1;A_1)\in O_2$ and let E_1,E_2,E_3 (resp. $\hat{E}_1,\hat{E}_2,\hat{E}_3$) be an orthonormal oriented frame of (M,g) defined on an open set $U\ni x_1$ (resp. $\hat{U}\ni\hat{x}_1$) such that $U\times\hat{U}\subset\pi_Q(O_2)$ and that $\star E_1,\star E_2,\star E_3$ (resp. $\hat{\star}\hat{E}_1,\hat{\star}\hat{E}_2,\hat{\star}\hat{E}_3$) are eigenvectors with eigenvalues $-K_1(\cdot),-K_2(\cdot),-K(\cdot)$ (resp. $-\hat{K}_1(\cdot),-\hat{K}_2(\cdot),-\hat{K}_3(\cdot)$) on U (resp. $\hat{U}$) as given by Proposition 5.6. Since $-K$ has multiplicity two on U (resp. $-\hat{K}$ has multiplicity twoon $\hat{U}$), we assume that $K_1(\cdot)=K(\cdot)\neq K_2(\cdot)$ everywhere on U, (resp. $\hat{K}_1(\cdot)=\hat{K}(\cdot)\neq\hat{K}_2(\cdot)$ everywhere on $\hat{U}$) without loss of generality. Recall that $K(x)=\hat{K}(\hat{x})$ for all $q=(x,\hat{x};A)\in O_2$ by Proposition 5.6 (and the remark that follows it) and hence for all $x\in U$, $\hat{x}\in\hat{U}$, $K(x)=\hat{K}(\hat{x})$. Taking $U,\hat{U}$ to be connected, this immediately implies that both K and $\hat{K}$ are constant functions on U and $\hat{U}$. We denote the common constant value simply by K.

Let X_A,Y_A,Z_A be chosen as in Proposition 5.6 for every $q=(x,\hat{x};A)\in O_2$. Then, since $\star Y_A$ is a unit eigenvector of $R|_x$ corresponding to the single eigenvalue $-K_2(x)$, we must have $E_2|_x=\pm Y_A$ and since $\nu(A\star Y_A)|_q$ is tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, by Lemma 5.9, it follows that for every $q=(x,\hat{x};A)\in O_2$, the vector $\nu(A\star E_2|_x)|_q$ is tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. This, together with Proposition 5.3, proves the claim for (M,g). Symmetrically (working in $Q(\hat{M},M)$) the claim also holds for $(\hat{M},\hat{g})$. The proof is complete. $\square$

We finally aim at proving that, using the notations of the previous lemma, $\Gamma^1_{(2,3)}(x)=\hat{\Gamma}^1_{(2,3)}(\hat{x})$ for all $(x,\hat{x})\in\pi_Q(O_2')$, where $O_2'=\pi_Q^{-1}(U\times\hat{U})\cap O_2$ and $U,\hat{U}$ are the domains of definition of orthonormal frames E_1,E_2,E_3 and $\hat{E}_1,\hat{E}_2,\hat{E}_3$ as given by Lemma 5.17 above.

To this end, we define $\theta:O_2'\to\mathbb{R}$ (restricting to smaller sets U, $\hat{U}$ if necessary) to be a smooth function such that for all $q=(x,\hat{x};A)\in O_2'$,

$$
\begin{aligned}
X_A=&\cos(\theta(q))E_1+\sin(\theta(q))E_3,\\
Z_A=&-\sin(\theta(q))E_1+\cos(\theta(q))E_3,
\end{aligned}
$$

where X_A,Z_A (and also Y_A) are chosen using Proposition 5.6. Indeed, this is well defined since X_A,Z_A lie in the plane $Y_A^\perp=E_2|_x^\perp$ as do also $E_1|_x,E_3|_x$, for all $q=(x,\hat{x};A)\in O_2'$. To simplify the notation, we write $c_\theta:=\cos(\theta(q))$ and $s_\theta:=\sin(\theta(q))$ as well as $\Gamma^i_{(j,k)}=\Gamma^i_{(j,k)}(x)$, when there is no room for confusion. We will be always working on O_2' if not mentioned otherwise. Moreover, it is convenient to denote the vector field E_2 of M by Y in the computations that follow (since $E_2|_x$ is parallel to Y_A for all $q\in O_2'$, this notation is justified). We will do computations on the "side of M" but the results are, by symmetry, always valid for $\hat{M}$ as well. We will make use of the following formulas which are easily verified (see Lemma 3.33),

$$
\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q X_{(\cdot)} &= (\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta - c_\theta\Gamma^1_{(3,1)} - s_\theta\Gamma^3_{(3,1)})Z_A + \Gamma^1_{(1,2)}Y,\\
\mathscr{L}_{\mathrm{R}}(Y)|_q X_{(\cdot)} &= (\mathscr{L}_{\mathrm{R}}(Y)|_q\theta - \Gamma^2_{(3,1)})Z_A,\\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q X_{(\cdot)} &= (\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta + s_\theta\Gamma^1_{(3,1)} - c_\theta\Gamma^3_{(3,1)})Z_A + \Gamma^1_{(2,3)}Y,\\
\mathscr{L}_{\mathrm{R}}(X_A)|_q Y &= -\Gamma^1_{(1,2)}X_A + \Gamma^1_{(2,3)}Z_A,\\
\mathscr{L}_{\mathrm{R}}(Y)|_q Y &= 0,\\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q Y &= -\Gamma^1_{(2,3)}X_A - \Gamma^1_{(1,2)}Z_A,\\
\mathscr{L}_{\mathrm{R}}(X_A)|_q Z_{(\cdot)} &= (-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})X_A - \Gamma^1_{(2,3)}Y,\\
\mathscr{L}_{\mathrm{R}}(Y)|_q Z_{(\cdot)} &= (-\mathscr{L}_{\mathrm{R}}(Y)|_q\theta + \Gamma^2_{(3,1)})X_A,\\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q Z_{(\cdot)} &= (-\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})X_A + \Gamma^1_{(1,2)}Y. \qquad (45)
\end{aligned}
$$

Remark 5.18 Notice that $\nu(A\star Z_A)|_q$ is not tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for any $q=(x,\hat{x};A)\in O_2'$. Indeed, otherwise there would be an open neighbourhood $O\subset O_2'$ of q such that for all $q'=(x',\hat{x}';A')$ the vectors $\nu(A'\star X_{A'})|_{q'},\nu(A'\star Y)|_{q'},\nu(A'\star Z_{A'})|_{q'}$ would span $V|_{q'}(\pi_Q)$ while being tangent to $T|_{q'}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, which implies $V|_{q'}(\pi_Q)\subset T|_{q'}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Then Corollary 4.18 would imply that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is open, which is not the case. We will use this fact frequently in what follows.

Taking U, $\hat{U}$ smaller if necessary, we may also assume that θ is actually defined not only on O_2' but on an open neighbourhood $\tilde{O}_2'$ of O_2 in Q. We will make this technical assumption to be able to write e.g. $\nu(A\star Z_A)|_q\theta$ whenever needed.

Lemma 5.19 For every $q=(x,\hat{x};A)\in O_2'$ we have

$$
\begin{aligned}
\nu(A\star Y)|_q\theta &= 1,\\
\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta &= c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)},\\
\mathscr{L}_{\mathrm{R}}(Y)|_q\theta &= \Gamma^2_{(3,1)} - \Gamma^1_{(2,3)}.
\end{aligned}
$$

Moreover, if one defines for $q=(x,\hat{x};A)\in O_2'$,

$$
\begin{aligned}
F_X|_q &:= \mathscr{L}_{\mathrm{NS}}(X_A)|_q - \Gamma^1_{(1,2)}\nu(A\star Z_A)|_q,\\
F_Z|_q &:= \mathscr{L}_{\mathrm{NS}}(Z_A)|_q - \Gamma^1_{(2,3)}\nu(A\star Z_A)|_q,
\end{aligned}
$$

then F_X,F_Z are smooth vector fields on O_2' tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$.

Proof. We begin by showing that $\nu(A\star Y)|_q\theta=1$. Indeed, we have for every $q=(x,\hat{x};A)\in O_2'$ that $\hat{g}(AZ_A,\hat{E}_2)=0$. Differentiating this w.r.t. $\nu(A\star Y)|_q$ yields

$$
0=\hat{g}(A(\star Y)Z_A,\hat{E}_2)-\nu(A\star Y)|_q\theta\hat{g}(AX_A,\hat{E}_2)=\hat{g}(AX_A,\hat{E}_2)(1-\nu(A\star Y)|_q\theta).
$$

We show that $\hat{g}(AX_A,\hat{E}_2)\neq 0$, whence $\nu(A\star Y)|_q\theta=1$. Indeed, if it were the case, then $AX_A\in E_2^{\perp}$ and hence $\hat{\star}(AX_A)$ would be an eigenvector of $\hat{R}|_{\hat{x}}$ with eigenvalue $-K$. This would then imply that

$$
\widetilde{\mathsf{Rol}}_q(\star X_A)=R(\star X_A)-A^{\overline{T}}\hat{R}(\hat{\star}(AX_A))A=-K\star X_A+KA^{\overline{T}}(\hat{\star}(AX_A)A=0.
$$

Because, $\widetilde{\mathsf{Rol}}_q(X_A \wedge Y) = 0$ as well, we see that $\widetilde{\mathsf{Rol}}_q$ has rank ≤ 1 as a map $\wedge^2 T|_x M \to \wedge^2 T|_x M$, which is a contradiction since $q \in O_2' \subset O_2$ and O_2 is, by definition, the set of points of the orbit where $\widetilde{\mathsf{Rol}}_q$ has rank 2. This contradiction establishes the above claim.

Next we compute the Lie brackets

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(Y), \nu((\cdot) \star X_{(\cdot)})]|_q =& - \mathscr{L}_{\mathrm{NS}}(A(\star X_A)Y)|_q + \nu(A \star \mathscr{L}_{\mathrm{R}}(Y)|_q X_{(\cdot)})|_q \\
=& - \mathscr{L}_{\mathrm{NS}}(AZ_A)|_q + (\mathscr{L}_{\mathrm{R}}(Y)|_q\theta - \Gamma^2_{(3,1)})\nu(A \star Z_A)|_q, \\
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot) \star Y)]|_q =& - \mathscr{L}_{\mathrm{R}}(\nu(A \star Y)|_q X_{(\cdot)})|_q - \nu(A \star Y)|_q \theta \mathscr{L}_{\mathrm{NS}}(A(\star Y)X_A)|_q \\
&+ \nu(A \star (c_\theta(-\Gamma^1_{(1,2)}E_1 + \Gamma^1_{(2,3)}E_3) + s_\theta(-\Gamma^1_{(2,3)}E_1 - \Gamma^1_{(1,2)}E_3)))|_q \\
=& - \mathscr{L}_{\mathrm{R}}(\nu(A \star Y)|_q X_{(\cdot)})|_q + \mathscr{L}_{\mathrm{NS}}(AZ_A)|_q \\
&- \Gamma^1_{(1,2)}\nu(A \star X_A)|_q + \Gamma^1_{(2,3)}\nu(A \star Z_A)|_q,
\end{aligned}$$

which sum is equal to

$$\begin{aligned}
&[\mathscr{L}_{\mathrm{R}}(Y), \nu((\cdot) \star X_{(\cdot)})]|_q + [\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot) \star Y)]|_q \\
=&(\mathscr{L}_{\mathrm{R}}(Y)|_q\theta - \Gamma^2_{(3,1)} + \Gamma^1_{(2,3)})\nu(A \star Z_A)|_q - \mathscr{L}_{\mathrm{R}}(\nu(A \star Y)|_q X_{(\cdot)})|_q \\
&- \Gamma^1_{(1,2)}\nu(A \star X_A)|_q.
\end{aligned}$$

Since this has to be tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, we get that the $\nu(A \star Z_A)|_q$-component vanished i.e.,

$$\mathscr{L}_{\mathrm{R}}(Y)|_q\theta = \Gamma^2_{(3,1)} - \Gamma^1_{(2,3)}.$$

Next compute

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot) \star X_{(\cdot)})]|_q =& - \nu(A \star X_A)|_q \theta \mathscr{L}_{\mathrm{R}}(Z_A)|_q - \mathscr{L}_{\mathrm{NS}}(A \underbrace{(\star X_A)X_A}_{=0})|_q \\
&+ \nu\big(A \star \big((\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta - c_\theta\Gamma^1_{(3,1)} - s_\theta\Gamma^3_{(3,1)})Z_A) + \Gamma^1_{(1,2)}Y\big)\big|_q,
\end{aligned}$$

and so we must have again that the $\nu(A \star Z_A)|_q$-component is zero i.e.,

$$\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta = c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}.$$

Since $\mathscr{L}_{\mathrm{NS}}(AZ_A)|_q = \mathscr{L}_{\mathrm{R}}(Z_A)|_q - \mathscr{L}_{\mathrm{NS}}(Z_A)|_q$, $[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot) \star Y)]|_q$ can be written as

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \nu((\cdot) \star Y)]|_q =& - F_Z|_q + \mathscr{L}_{\mathrm{R}}(Z_A)|_q - \mathscr{L}_{\mathrm{R}}(\nu(A \star Y)|_q X_{(\cdot)})|_q - \Gamma^1_{(1,2)}\nu(A \star X_A)|_q \\
=& - F_Z|_q - \Gamma^1_{(1,2)}\nu(A \star X_A)|_q,
\end{aligned}$$

which proves that F_Z, as defined in the statement, is indeed tangent to the orbit on O_2'. To show that F_X is also tangent to the orbit we compute

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}), \nu((\cdot) \star Y)]|_q =& - \mathscr{L}_{\mathrm{R}}(\nu(A \star Y)|_q Z_{(\cdot)})|_q - \nu(A \star Y)|_q \theta \mathscr{L}_{\mathrm{NS}}(A(\star Y)Z_A)|_q \\
&+ \nu(A \star (-s_\theta(-\Gamma^1_{(1,2)}E_1 + \Gamma^1_{(2,3)}E_3) + c_\theta(-\Gamma^1_{(2,3)}E_1 - \Gamma^1_{(1,2)}E_3)))|_q \\
=& - \mathscr{L}_{\mathrm{R}}(\nu(A \star Y)|_q Z_{(\cdot)})|_q - \mathscr{L}_{\mathrm{NS}}(AX_A)|_q \\
&- \Gamma^1_{(1,2)}\nu(A \star Z_A)|_q - \Gamma^1_{(2,3)}\nu(A \star X_A)|_q \\
=&F_X|_q - \mathscr{L}_{\mathrm{R}}(X_A)|_q - \mathscr{L}_{\mathrm{R}}(\nu(A \star Y)|_q Z_{(\cdot)})|_q - \Gamma^1_{(2,3)}\nu(A \star X_A)|_q \\
=&F_X|_q - \Gamma^1_{(2,3)}\nu(A \star X_A)|_q,
\end{aligned}$$

which finishes the proof. $\square$

Lemma 5.20 For all $(x, \hat{x}) \in \pi_Q(O_2')$ one has

$$\Gamma^1_{(2,3)}(x) = \hat{\Gamma}^1_{(2,3)}(\hat{x}).$$

Proof. We begin by observing that for all $q = (x, \hat{x}; A) \in O_2'$ one has $\hat{g}(AZ_A, \hat{E}_2) = 0$. Indeed, AZ_A and $\hat{E}_2|_{\hat{x}}$ are eigenvectors of $\hat{R}|_{\hat{x}}$ corresponding to non-equal eigenvalues $-K$ and $-\hat{K}_2(\hat{x})$, hence they must be orthogonal. Since $AZ_A \in \hat{E}_2|_{\hat{x}}^{\perp}$, there is a $\hat{\theta} = \hat{\theta}(q)$, for all $q = (x, \hat{x}; A) \in O_2'$, such that

$$AZ_A = -s_{\hat{\theta}}\hat{E}_1 + c_{\hat{\theta}}\hat{E}_3.$$

Because $AX_A, AY \in (AZ_A)^{\perp}$, there exits also a $\hat{\phi} = \hat{\phi}(q)$ such that

$$\begin{aligned} AX_A =& c_{\hat{\phi}}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + s_{\hat{\phi}}\hat{E}_2, \\ AY =& - s_{\hat{\phi}}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + c_{\hat{\phi}}\hat{E}_2. \end{aligned}$$

Moreover, Lemma 5.19 along with Eq. (45) implies that $\mathscr{L}_{\mathrm{R}}(Y)|_q Z_{(\cdot)}$ simplifies to

$$\mathscr{L}_{\mathrm{R}}(Y)|_q Z_{(\cdot)} = \Gamma^1_{(2,3)}(x) X_A.$$

Therefore, differentiating $\hat{g}(AZ_A, \hat{E}_2) = 0$ with respect to $\mathscr{L}_{\mathrm{R}}(X_A)|_q$, one obtains

$$\begin{aligned} 0 =& \mathscr{L}_{\mathrm{R}}(Y)|_q \hat{g}((\cdot)Z_{(\cdot)}, \hat{E}_2) = \hat{g}(A\mathscr{L}_{\mathrm{R}}(Y)|_q Z_{(\cdot)}, \hat{E}_2) + \hat{g}(AZ_A, \hat{\nabla}_{AY}\hat{E}_2) \\ =& \Gamma^1_{(2,3)}\hat{g}(AX_A, \hat{E}_2) + \hat{g}\big(AZ_A, -s_{\hat{\phi}}c_{\hat{\theta}}(-\hat{\Gamma}^1_{(1,2)}\hat{E}_1 + \hat{\Gamma}^1_{(2,3)}\hat{E}_3) - s_{\hat{\phi}}s_{\hat{\theta}}(-\hat{\Gamma}^1_{(2,3)}\hat{E}_1 - \hat{\Gamma}^1_{(1,2)}\hat{E}_3)\big) \\ =& s_{\hat{\phi}}\Gamma^1_{(2,3)} - s_{\hat{\phi}}\hat{g}\big(AZ_A, \hat{\Gamma}^1_{(2,3)}AZ_A - \hat{\Gamma}^1_{(1,2)}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3)\big) \\ =& s_{\hat{\phi}}(\Gamma^1_{(2,3)}(x) - \hat{\Gamma}^1_{(2,3)}(\hat{x})). \end{aligned}$$

We claim that $\sin(\hat{\phi}(q)) \neq 0$ for $q = (x, \hat{x}; A) \in O_2'$, which implies that $\Gamma^1_{(2,3)}(x) - \hat{\Gamma}^1_{(2,3)}(\hat{x}) = 0$ and finishes the proof. Indeed, $\sin(\hat{\phi}(q)) = 0$ would mean that $AX_A = \pm(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3)$, thus $AX_A \in \hat{E}_2^{\perp}$. By the argument at the beginning of the proof of Lemma 5.19, this would be a contradiction. □

Corollary 5.21 The following holds.

(i) If for some $(x_1, \hat{x}_1) \in \pi_Q(O_2')$, one has $\Gamma^1_{(2,3)}(x_1) \neq 0$ (or $\hat{\Gamma}^1_{(2,3)}(\hat{x}_1) \neq 0$), there are open neighbourhoods $U' \ni x_1$, $\hat{U}' \ni \hat{x}_1$ such that (U', g), $(\hat{U}', \hat{g})$ are both of class $\mathcal{M}_\beta$ for $\beta = \Gamma^1_{(2,3)}(x_1)$ (or $\beta = \hat{\Gamma}^1_{(2,3)}(\hat{x}_1)$).

(ii) If for some $(x_1, \hat{x}_1) \in \pi_Q(O_2')$, one has $\Gamma^1_{(2,3)}(x_1) = 0$ (or $\hat{\Gamma}^1_{(2,3)}(\hat{x}_1) = 0$), there are open neighbourhoods $U' \ni x_1$, $\hat{U}' \ni \hat{x}_1$ such that $U' \times \hat{U}' \subset \pi_Q(O_2')$ and isometries $F : (I \times N, h_f) \to (U, g)$, $\hat{F} : (I \times \hat{N}, \hat{h}_{\hat{f}}) \to (\hat{U}, \hat{g})$, where $I \subset \mathbb{R}$ is an open interval, such that

$$\frac{f''(t)}{f(t)} = -K = \frac{\hat{f}''(t)}{\hat{f}(t)}, \quad \forall t \in I.$$

Proof. Let $U', \hat{U}'$ be connected neighbourhoods of $x_1, \hat{x}_1$ such that $U' \times \hat{U}' \subset \pi_Q(O_2')$ (recall that by Lemma 5.9, $\pi_Q(O_2')$ is open in $M \times \hat{M}$).

(i) Set $\beta = \Gamma^1_{(2,3)}(x_1) \neq 0$. By Lemma 5.20, one has for every $x \in U'$, $\hat{x} \in \hat{U}'$ that

$$\begin{aligned}\hat{\Gamma}^1_{(2,3)}(\hat{x}) &= \Gamma^1_{(2,3)}(x_1) = \beta, \\ \Gamma^1_{(2,3)}(x) &= \hat{\Gamma}^1_{(2,3)}(\hat{x}_1) = \beta.\end{aligned}$$

By Proposition C.17 case (ii), it follows that (after shrinking $U', \hat{U}'$) (U, g) and $(\hat{U}, \hat{g})$ are both of class $\mathcal{M}_\beta$ This gives (i).

(ii) By Lemma 5.20, one has for every $x \in U'$, $\hat{x} \in \hat{U}'$ that

$$\begin{aligned}\hat{\Gamma}^1_{(2,3)}(\hat{x}) &= \Gamma^1_{(2,3)}(x_1) = 0, \\ \Gamma^1_{(2,3)}(x) &= \hat{\Gamma}^1_{(2,3)}(\hat{x}_1) = 0,\end{aligned}$$

i.e. $\Gamma^1_{(2,3)}$ and $\hat{\Gamma}^1_{(2,3)}$ vanish on $U', \hat{U}'$, respectively. Then Proposition C.17 case (iii) gives (after shrinking $U', \hat{U}'$) the desired isometries $F, \hat{F}$. Moreover, Eq. (57) in that proposition gives, since $E_2 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{\partial}{\partial r}$,

$$\begin{aligned}-K &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{f'(r)}{f(r)} + \Big(-\frac{f'(r)}{f(r)}\Big) - 0^2 = \frac{f''(r)}{f(r)}, \\ -K &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{\hat{f}'(r)}{\hat{f}(r)} + \Big(-\frac{\hat{f}'(r)}{\hat{f}(r)}\Big) - 0^2 = \frac{\hat{f}''(r)}{\hat{f}(r)},\end{aligned}$$

where $r \in I$. This proves (ii). □

5.2.2 Local Structures for the Manifolds Around $q \in O_1$

In analogy to Proposition (5.6) we will first prove the following result. In the results below that concern O_1, we always assume that $O_1 \neq \emptyset$. For the next proposition, contrary to an analogous Proposition 5.6 of Subsubsection 5.2.1, we do not need to assume that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open. The subsequent result only relies on the fact that O_1 is not empty.

Proposition 5.22 Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. Then for every $q = (x, \hat{x}; A) \in O_1$ there exist an orthonormal pair $X_A, Y_A \in T|_x M$ such that if $Z_A := \star(X_A \wedge Y_A)$ then X_A, Y_A, Z_A is a positively oriented orthonormal pair with respect to which R and $\widetilde{\mathrm{Rol}}$

may be written as

$$
\begin{aligned}
R(X_A \wedge Y_A) &= \begin{pmatrix} 0 & K(x) & 0 \\ -K(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \star R(X_A \wedge Y_A) = \begin{pmatrix} 0 \\ 0 \\ -K(x) \end{pmatrix}, \\
R(Y_A \wedge Z_A) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K(x) \\ 0 & -K(x) & 0 \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K(x) \\ 0 \\ 0 \end{pmatrix}, \\
R(Z_A \wedge X_A) &= \begin{pmatrix} 0 & 0 & -K_2(x) \\ 0 & 0 & 0 \\ K_2(x) & 0 & 0 \end{pmatrix}, \quad \star R(Z_A \wedge X_A) = \begin{pmatrix} 0 \\ -K_2(x) \\ 0 \end{pmatrix}, \\
\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A) &= 0, \\
\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) &= 0, \\
\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) &= \begin{pmatrix} 0 & 0 & -K_2^{\mathsf{Rol}}(q) \\ 0 & 0 & 0 \\ K_2^{\mathsf{Rol}}(q) & 0 & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} 0 \\ -K_2^{\mathsf{Rol}}(q) \\ 0 \end{pmatrix},
\end{aligned} \tag{46}
$$

where K, K_2 are real-valued functions defined on M.

With respect to X_A, Y_A, Z_A given by the theorem, we also have

$$
\begin{aligned}
\star A^T \hat{R}(AX_A \wedge AY_A)A &= \begin{pmatrix} 0 \\ 0 \\ -K(x) \end{pmatrix}, \\
\star A^T \hat{R}(AY_A \wedge AZ_A)A &= \begin{pmatrix} -K(x) \\ 0 \\ 0 \end{pmatrix}, \\
\star A^T \hat{R}(AZ_A \wedge AX_A)A &= \begin{pmatrix} 0 \\ -K_2(x) + K_2^{\mathsf{Rol}}(q) \\ 0 \end{pmatrix}.
\end{aligned} \tag{47}
$$

Relevant observations regarding the previous proposition are collected next.

Remark 5.23 (a) The last proposition says that $\star X_A, \star Y_A, \star Z_A$ are eigenvectors of $R|_x$, for every $q = (x, \hat{x}; A) \in O_1$, with corresponding eigenvalues $-K(x)$, $-K_2(x)$ and $-K(x)$. Changing the roles of (M, g) and $(\hat{M}, \hat{g})$, the proposition gives that, for every $q = (x, \hat{x}; A) \in O_1$, eigenvectors $\hat{\star}\hat{X}_A, \hat{\star}\hat{Y}_A, \hat{\star}\hat{Z}_A$ are eigenvectors of $\hat{R}|_x$, with corresponding eigenvalues $-\hat{K}(\hat{x})$, $-\hat{K}_2(\hat{x})$ and $-\hat{K}(x)$.

(b) The eigenvalues K and $\hat{K}$ coincide on the set of points that can be reached, locally, by the rolling. More precisely, Proposition 5.22 tells us that

$$
-\hat{K}(\hat{x}) = -K(x), \quad \forall (x, \hat{x}) \in \pi_Q(O_1),
$$

and that this eigenvalue is at least a double eigenvalue for both $R|_x$ and $\hat{R}|_{\hat{x}}$.

(c) The above at-least-double eigenvalue cannot be a triple eigenvalue for both $R|_x$ and $\hat{R}|_{\hat{x}}$ at the same time, for $(x, \hat{x}) \in \pi_Q(O_1)$. Indeed, if $K_2(x) = K(x)$ and $\hat{K}_2(\hat{x}) = \hat{K}(\hat{x})$, then clearly this would imply that $\mathsf{Rol}_q = 0$, which contradicts the fact that $q \in O_1$ implies $\operatorname{rank} \mathsf{Rol}_q = 1$.

(d) It is not clear that the assignments $q \mapsto X_A, Z_A$ can be made locally smoothly on O_1. However, it is the case for the assignment $q \mapsto Y_A$. In addition, for every $q = (x, \hat{x}; A) \in O_1$, the choice of Y_A and $\hat{Y}_A$ are uniquely determined up to multiplication by -1. Indeed, $\star Y_A = Z_A \wedge X_A$ is a unit eigenvector of $\widetilde{\mathsf{Rol}}_q$ corresponding to the simple non-zero eigenvalue $-K_2^{\mathsf{Rol}}(q)$ (it is non-zero since $\operatorname{rank} \mathsf{Rol}_q = 1$, $q \in O_1$). By symmetry, the same holds of $\hat{Y}_A$ as well. Then

$$AY_A = \pm \hat{Y}_A, \quad \forall q = (x, \hat{x}; A) \in O_1.$$

We begin by the following simple lemma.

Lemma 5.24 For every $q = (x, \hat{x}; A) \in O_1$ and any orthonormal pair (which exists) $X_A, Y_A \in T|_x M$ such that $X_A, Y_A, Z_A := \star(X_A \wedge Y_A)$ is an oriented orthonormal basis of $T|_x M$ and $\mathsf{Rol}_q(X_A \wedge Y_A) = 0$, $\mathsf{Rol}_q(Y_A \wedge Z_A) = 0$, one has with respect to the basis X_A, Y_A, Z_A,

$$R(X_A \wedge Y_A) = \begin{pmatrix} 0 & K_A & \alpha_A \\ -K_A & 0 & -\beta_A \\ -\alpha_A & \beta_A & 0 \end{pmatrix}, \quad \star R(X_A \wedge Y_A) = \begin{pmatrix} \beta_A \\ \alpha_A \\ -K_A \end{pmatrix},$$

$$R(Y_A \wedge Z_A) = \begin{pmatrix} 0 & -\beta_A & \xi_A \\ \beta_A & 0 & K_A^1 \\ -\xi_A & -K_A^1 & 0 \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_A^1 \\ \xi_A \\ \beta_A \end{pmatrix},$$

$$R(Z_A \wedge X_A) = \begin{pmatrix} 0 & -\alpha_A & -K_A^2 \\ \alpha_A & 0 & -\xi_A \\ K_A^2 & \xi_A & 0 \end{pmatrix}, \quad \star R(Z_A \wedge X_A) = \begin{pmatrix} \xi_A \\ -K_A^2 \\ \alpha_A \end{pmatrix},$$

$$\widetilde{\mathsf{Rol}}_q(X_A \wedge Y_A) = 0,$$

$$\widetilde{\mathsf{Rol}}_q(Y_A \wedge Z_A) = 0,$$

$$\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} 0 & 0 & -K_2^{\mathsf{Rol}} \\ 0 & 0 & 0 \\ K_2^{\mathsf{Rol}} & 0 & 0 \end{pmatrix}, \quad \star\widetilde{\mathsf{Rol}}_q(Z_A \wedge X_A) = \begin{pmatrix} 0 \\ -K_2^{\mathsf{Rol}} \\ 0 \end{pmatrix}.$$

Moreover, the choice of the above quantities can be made locally smoothly on O_1.

Proof. We only need to prove the existence of an oriented orthonormal basis X_A, Y_A and Z_A such that $\mathsf{Rol}_q(X_A \wedge Y_A) = 0$, $\mathsf{Rol}_q(Y_A \wedge Z_A) = 0$. Indeed, when this has been established, one may use Lemma 5.8, where we now have $K_1^{\mathsf{Rol}}(q) = 0$, $\alpha(q) = 0$ because $\mathsf{Rol}_q(Y_A \wedge Z_A) = 0$, to conclude.

Since for a given $q = (x, \hat{x}; A) \in O_1$, $\widetilde{\mathsf{Rol}}_q : \wedge^2 T|_x M \to \wedge^2 T|_x M$ is symmetric linear map that has rank 1, it follows that its eigenspaces are orthogonal and its kernel has dimension exactly 2. Thus there is an orthonormal basis $\omega_1, \omega_2, \lambda$ of $\wedge^2 T|_x M$ such that $\widetilde{\mathsf{Rol}}_q(\omega_i) = 0$, $i = 1, 2$. Taking $X_A = \star\omega_1$, $Z_A = \star\omega_2$ and $Y_A = \star\lambda$ we get, up to replacing X_A with $-X_A$ if necessary, an oriented orthonormal basis of $T|_x M$ such that $\mathsf{Rol}(X_A \wedge Y_A) = 0$, $\mathsf{Rol}(Y_A \wedge Z_A) = 0$. $\square$

As a consequence of the lemma and because $A^T \hat{R}(AX, AY)A = R(X, Y) - \widetilde{\mathsf{Rol}}_q(X, Y)$ for $X, Y \in T|_x M$, we have that w.r.t. the oriented orthonormal basis

X_A, Y_A, Z_A,

$$\star A^{\overline{T}}\hat{R}(AX_A, AY_A)A = \begin{pmatrix} \beta_A \\ \alpha_A \\ -K_A \end{pmatrix},$$
$$\star A^{\overline{T}}\hat{R}(AY_A, AZ_A)A = \begin{pmatrix} -K_A^1 \\ \xi_A \\ \beta_A \end{pmatrix},$$
$$\star A^{\overline{T}}\hat{R}(AZ_A, AX_A)A = \begin{pmatrix} \xi_A \\ -K_A^2 + K_2^{\mathsf{Rol}} \\ \alpha_A \end{pmatrix}. \tag{48}$$

The assumption that rank $\mathsf{Rol}_q = 1$ is equivalent to the fact that for every $q = (x, \hat{x}; A) \in O_1$,

$$K_2^{\mathsf{Rol}}(q) \neq 0. \tag{49}$$

This implies that Y_A is uniquely determined up to multiplication by -1 (see also Remark 5.23 above). Hence, in particular, for every $q = (x, \hat{x}; A) \in O_1$,

$$\mathsf{Rol}_q(\wedge^2 TM)(A) = \mathrm{span}\{\nu(A(Z_A \wedge X_A))|_q\} = \mathrm{span}\{\nu(A \star Y_A)|_q\}.$$

We will now show that, with any (non-unique) choice of a pair X_A, Y_A as in Lemma 5.24, one has that $\alpha_A = 0$ and $K_A = K_A^1$.

Lemma 5.25 If one chooses any $X_A, Y_A, Z_A = \star(X_A \wedge Y_A)$ as in Lemma 5.24, then

$$\beta_A = 0, \quad K_A = K_A^1, \quad \forall q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0).$$

Proof. Fix $q = (x, \hat{x}; A) \in O_1$. Choosing in Corollary 4.14 $X, Y \in \mathrm{VF}(M)$ such that $X|_x = X_A$, $Y|_x = Y_A$, we get, since $\mathsf{Rol}_q(X_A \wedge Y_A) = 0$,

$$\begin{aligned}
&\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X, Y)(\cdot)), \nu(\mathsf{Rol}(Z, W)(\cdot))\big]\big|_q \\
=&A\big[R(X_A \wedge Y_A), R(Z|_x \wedge W|_x)\big]_{\mathfrak{so}} - \big[\hat{R}(AX_A \wedge AY_A), \hat{R}(AZ|_x \wedge AW|_x)\big]_{\mathfrak{so}}A \\
&+ \hat{R}(AX_A \wedge A\widetilde{\mathsf{Rol}}_q(Z|_x \wedge W|_x)Y_A)A + \hat{R}(A\widetilde{\mathsf{Rol}}_q(Z|_x \wedge W|_x)X_A, AY_A)A.
\end{aligned}$$

Since $q' = (x', \hat{x}'; A') \mapsto \nu(\mathsf{Rol}(\wedge^2 T|_{x'}M)(A'))|_{q'} = \mathrm{span}\{\nu(A' \star Y_{A'})\}$ is a smooth rank one distribution on O_1, it follows that it is involutive and hence for all $X, Y, Z, W \in \mathrm{VF}(M)$,

$$\big[\nu(\mathsf{Rol}(X \wedge Y)(\cdot)), \nu(\mathsf{Rol}(Z \wedge W)(\cdot))\big]\big|_q \in \mathrm{span}\{\nu(A \star Y_A)|_q\},$$

where we used that $\mathsf{Rol}(\wedge^2 TM)(A) = \mathrm{span}\{A \star Y_A\}$ as observed above.

We compute the right hand side of this formula in different cases. We begin by taking any smooth vector fields X, Y, Z, W with $X|_x = X_A$, $Y|_x = Y_A$, $Z|_x = Z_A$,

$W|_x = X_A$. One gets

$$
\begin{aligned}
&A^{\overline{T}}\nu|_q^{-1}\big[\nu(\mathsf{Rol}(X,Y)(\cdot)),\nu(\mathsf{Rol}(Z,W)(\cdot))\big]\big|_q \\
=&\big[R(X_A\wedge Y_A),R(Z_A\wedge X_A)\big]_{\mathfrak{so}}-\big[A^{\overline{T}}\hat{R}(AX_A\wedge AY_A)A,A^{\overline{T}}\hat{R}(AZ_A\wedge AX_A)A\big]_{\mathfrak{so}} \\
&+A^{\overline{T}}\hat{R}(AX_A\wedge \mathsf{Rol}(Z_A\wedge X_A)(A)Y_A)A+A^{\overline{T}}\hat{R}(\mathsf{Rol}(Z_A\wedge X_A)(A)X_A\wedge AY_A)A \\
=&\begin{pmatrix}\beta_A\\ \alpha_A\\ -K_A\end{pmatrix}\wedge\begin{pmatrix}0\\ -\tilde{K}_2^{\mathsf{Rol}}\\ 0\end{pmatrix}+A^{\overline{T}}\hat{R}(AX_A\wedge 0)A+A^{\overline{T}}\hat{R}(\tilde{K}_2^{\mathsf{Rol}}AZ_A\wedge AY_A)A \\
=&\begin{pmatrix}-K_A\tilde{K}_2^{\mathsf{Rol}}\\ 0\\ -\beta_A\tilde{K}_2^{\mathsf{Rol}}\end{pmatrix}-\tilde{K}_2^{\mathsf{Rol}}\begin{pmatrix}-K_A^1\\ \xi_A\\ \beta_A\end{pmatrix}=\begin{pmatrix}\tilde{K}_2^{\mathsf{Rol}}(-K_A+K_A^1)\\ \tilde{K}_2^{\mathsf{Rol}}\xi_A\\ -2\beta_A\tilde{K}_2^{\mathsf{Rol}}\end{pmatrix}\in\operatorname{span}\{\nu(A\star Y_A)|_q\}.
\end{aligned}
$$

Because $\tilde{K}_2^{\mathsf{Rol}}(q)\neq 0$, this immediately implies that

$$-K_A+K_A^1=0,\quad \beta_A=0.$$

This completes the proof. □

We will now rotate X_A,Y_A,Z_A in such a way that we can set α_A equal to zero.

Lemma 5.26 For every $q=(x,\hat{x};A)\in O_1$ there are orthonormal $X_A,Y_A\in T|_xM$ such that $X_A,Y_A,Z_A=\star(X_A\wedge Y_A)$ is an oriented orthonormal basis of $T|_xM$ with respect to which in Lemma 5.24 one has $\alpha_A=0$.

Proof. Fix $q=(x,\hat{x};A)\in O_1$, choose any $X_A,Y_A,Z_A=\star(X_A\wedge Y_A)$ as in Lemma 5.24 and suppose $\alpha_A\neq 0$ (otherwise we are done). For $t\in\mathbb{R}$, set

$$\begin{pmatrix}X_A(t)\\ Z_A(t)\end{pmatrix}=\begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}X_A\\ Z_A\end{pmatrix}.$$

Then clearly $Y_A(t):=\star(X_A(t)\wedge Z_A(t))=\star(X_A\wedge Z_A)=Y_A$ and $X_A(t),Y_A(t),Z_A(t)$ is an orthonormal positively oriented basis of $T|_xM$. Since $\widetilde{\mathsf{Rol}}_q$ is a symmetric map $\wedge^2T|_xM\to\wedge^2T|_xM$ and since $\star X_A,\star Z_A$ are its eigenvectors corresponding to the eigenvalue 0, it follows that $\star X_A(t),\star Z_A(t)$, which are just rotated $\star X_A,\star Z_A$ in the plane that they span, are eigenvectors of Rol_q corresponding to the eigenvalue 0, i.e. $\mathsf{Rol}_q(X_A(t)\wedge Y_A)=0$, $\mathsf{Rol}_q(Y_A\wedge Z_A(t))=0$ for all $t\in\mathbb{R}$.

Hence the conclusion of Lemma 5.24 holds for the basis $X_A(t),Y_A,Z_A(t)$ and we write $\xi_A(t),\alpha_A(t)$, $\beta_A(t)$, $K_A(t)$, $K_A^1(t)$, $K_A^2(t)$ for the coefficients of the matrices of R given there w.r.t. $X_A(t),Y_A,Z_A(t)$. Then Lemma 5.25 implies that $\beta_A(t)=0$, $K_A(t)=K_A^1(t)$ for all $t\in\mathbb{R}$. We now compute

$$
\begin{aligned}
\alpha_A(t)=&g(R(X_A(t)\wedge Y_A)Z_A(t),X_A(t))=g(R(Z_A(t)\wedge X_A(t))X_A(t),Y_A(t))\\
=&-g(R(Z_A\wedge X_A)Y_A,X_A(t))\\
=&-g(-\alpha_AX_A+\xi_AZ_A,\cos(t)X_A+\sin(t)Z_A)\\
=&-\alpha_A\cos(t)+\xi_A\sin(t).
\end{aligned}
$$

Thus choosing $t_0\in\mathbb{R}$ such that

$$\cot(t_0)=\frac{\xi_A}{\alpha_A},$$

we get that $\alpha_A(t_0)=0$. As already observed, we also have $\beta_A(t_0)=0$, $K_A^1(t_0)=K_A(t_0)$ and $\mathsf{Rol}_q(X_A(t_0)\wedge Y_A)=0$, $\mathsf{Rol}_q(Y_A\wedge Z_A(t_0))=0$. □

Since α_A and β_A vanish w.r.t X_A, Y_A, Z_A, as chosen by the previous lemma, we have that $-K_A$ is an eigenvalue of $R|_x$ with eigenvector $X_A \wedge Y_A$, where $q = (x, \hat{x}; A) \in O_1$. Knowing this, we may prove that even ξ_A is zero as well and that (automatically) $-K_A$ is a at least a double eigenvalue of $R|_x$. This is given in the lemma that follows.

Lemma 5.27 If $q = (x, \hat{x}; A) \in O_1$ and X_A, Y_A, Z_A as in Lemma 5.26, then $\xi_A = 0$.

Proof. Since for any $q = (x, \hat{x}; A) \in O_1$, $-K_A$ is an eigenvalue of $R|_x$, we know that its value only depends on the point x of M and hence we consider it as a smooth function $-K(x)$ on M. We claim that that $-K(x)$ is at least a double eigenvalue of $R|_x$. Suppose it is not. Then in a neighbourhood U of x we have that $-K(y)$ is a simple eigenvalue of $R|_y$ for all $y \in U$. In that case, we may choose smooth vector fields X, Y on U, taking U smaller if necessary, such that $X|_y \wedge Y|_y$ is a (non-zero) eigenvector of $R|_y$ corresponding to $-K(y)$ and $X|_x = X_A$, $Y|_x = Y_A$. Write $O := \pi_{Q,M}^{-1}(U) \cap O_1$. For any $(y, \hat{y}; B) \in O$, we know that $X_B \wedge Y_B$ is a unit eigenvector of $R|_y$ corresponding to $-K(y)$ and hence, modulo replacing X by $-X$, we have $X_B \wedge Y_B = X|_y \wedge Y|_y$. Then, for all $(y, \hat{y}; B) \in O$ with $y \in U$, one has

$$\nu(\mathsf{Rol}(X|_y \wedge Y|_y)(B))|_{(y,\hat{y};B)} = \nu(\mathsf{Rol}(X_B \wedge Y_B)(B))|_{(y,\hat{y};B)} = 0,$$

i.e., $\nu(\mathsf{Rol}(X \wedge Y)(\cdot))$ is a zero vector field on the open subset O of the orbit. If we also take some smooth vector fields Z, W such that $Z|_x = Z_A$, $W|_x = X_A$, we get by the fact that $\nu(\mathsf{Rol}(X \wedge Y)(\cdot)) = 0$ and from the computations in the proof of Lemma 5.25 that

$$0 = \nu|_q^{-1}\big[\nu(\mathsf{Rol}(X, Y)(\cdot)), \nu(\mathsf{Rol}(Z, W)(\cdot))\big]\big|_q = \begin{pmatrix} K_2^{\mathsf{Rol}}(-K_A + K_A^1) \\ K_2^{\mathsf{Rol}}\xi_A \\ -2\beta_A K_2^{\mathsf{Rol}} \end{pmatrix} = \begin{pmatrix} 0 \\ K_2^{\mathsf{Rol}}\xi_A \\ 0 \end{pmatrix}.$$

Since $K_2^{\mathsf{Rol}}(q) \neq 0$ we get $\xi_A = 0$. This implies, along with the results obtained in the previous lemma (i.e. $K = K_A^1$, $\beta_A = \alpha_A = 0$), that w.r.t. the basis X_A, Y_A, Z_A, one has

$$\star R(X_A \wedge Y_A) = \begin{pmatrix} 0 \\ 0 \\ -K_A \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_A \\ 0 \\ 0 \end{pmatrix},$$

which means that $X_A \wedge Y_A$ and $Y_A \wedge Z_A$ are linearly independent eigenvectors of $R|_x$ corresponding to the eigenvalue $-K_A = -K(x)$. This is in contradiction to what we assumed at the beginning of the proof. Hence we have that $-K_A$ is, for every $q = (x, \hat{x}; A) \in O_1$, an eigenvalue of $R|_x$ of multiplicity at least 2. Finally, since we know that w.r.t. X_A, Y_A, Z_A,

$$\star R(X_A \wedge Y_A) = \begin{pmatrix} 0 \\ 0 \\ -K_A \end{pmatrix}, \quad \star R(Y_A \wedge Z_A) = \begin{pmatrix} -K_A \\ \xi_A \\ 0 \end{pmatrix}, \quad \star R(Z_A \wedge X_A) = \begin{pmatrix} \xi_A \\ -K_A^2 \\ 0 \end{pmatrix},$$

and since $R|_x$ is a symmetric linear map having double eigenvalue $-K_A$, then there exists a unit eigenvector ω of $R|_x$ corresponding to $-K_A$ which belongs to the plane orthogonal to $X_A \wedge Y_A$ (in $\wedge^2 T|_x M$). Hence, $\omega = \cos(t) Y_A \wedge Z_A + \sin(t) Z_A \wedge X_A$ for some $t \in \mathbb{R}$ and

$$
\begin{aligned}
-K_A \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} &= -K_A \star \omega = \star R(\omega) = \cos(t) \star R(Y_A \wedge Z_A) + \sin(t) \star R(Z_A \wedge X_A) \\
&= \cos(t) \begin{pmatrix} -K_A \\ \xi_A \\ 0 \end{pmatrix} + \sin(t) \begin{pmatrix} \xi_A \\ -K_A^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -K_A \cos(t) + \xi_A \sin(t) \\ \xi_A \cos(t) - K_A^2 \sin(t) \\ 0 \end{pmatrix},
\end{aligned}
$$

where the matrices are formed w.r.t. X_A, Y_A, Z_A. From the first row, we get $\xi_A \sin(t) = 0$. So either $\xi_A = 0$ and we are done or $\sin(t) = 0$, implying that $\omega = 1_\pm Y_A \wedge Z_A$ with $1_\pm \in \{-1, +1\}$ and hence

$$
\begin{pmatrix} -K_A \\ \xi_A \\ 0 \end{pmatrix} = \star R(Y_A \wedge Z_A) = 1_\pm \star R(\omega) = -K_A(1_\pm \star \omega) = -K_A \star (Y_A \wedge Z_A) = \begin{pmatrix} -K_A \\ 0 \\ 0 \end{pmatrix},
$$

which gives $\xi_A = 0$ anyhow.

□

The previous lemma implies Proposition 5.22, since now $-K_A = -K_A^1$, $-K_A^2$ are eigenvalues of $R|_x$ for every $(x, \hat{x}; A) \in O_1$ and hence, defining $K(x) := K_A$, $K_2(x) := K_A^2$, we obtain well defined functions $K, K_2 : M \to \mathbb{R}$.

The following Proposition is the last result of this subsection. Notice that it does need the assumption that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open while the previous results do not need this assumption.

Proposition 5.28 Suppose $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not open in Q. Then there is an open dense subset O_1° of O_1 such that for every $q_1 = (x_1, \hat{x}_1; A_1) \in O_1^\circ$ there are neighbourhoods U and $\hat{U}$ of x_1 and $\hat{x}_1$, respectively, such that either

(i) both $(U, g|_U)$, $(\hat{U}, \hat{g}|_{\hat{U}})$ are of class $\mathcal{M}_\beta$ or

(ii) both $(U, g|_U)$, $(\hat{U}, \hat{g}|_{\hat{U}})$ are isometric to warped products $(I \times N, h_f)$, $(I \times \hat{N}, \hat{h}_{\hat{f}})$ and $\frac{f'(r)}{f(r)} = \frac{\hat{f}'(r)}{\hat{f}(r)}$, for all $r \in I$.

Moreover, there is an oriented orthonormal frame E_1, E_2, E_3 (resp. $\hat{E}_1, \hat{E}_2, \hat{E}_3$) defined on U (resp. on $\hat{U}$) respectively, such that $\star E_1, \star E_3$ (resp. $\star \hat{E}_1, \star \hat{E}_3$) are eigenvectors of $\hat{R}$ with common eigenvalue $-K(\cdot)$ (resp. $-\hat{K}(\cdot)$) and one has

$$
A_1 E_2|_{x_1} = \hat{E}_2|_{\hat{x}_1}.
$$

Proof. Let $q_1 = (x_1, \hat{x}_1; A_1) \in O_1$. As observed in Remark 5.23, either $R|_{x_1}$ or $\hat{R}|_{\hat{x}_1}$ has $-K_2(x_1)$ or $-\hat{K}_2(\hat{x}_1)$, respectively, as a single eigenvalue. By symmetry of the problem in (M, g), $(\hat{M}, \hat{g})$, we assume that this is the case for $R|_{x_1}$. Hence there is a neighbourhood U of x_1 such that $K_2(x) \neq K(x)$ for all $x \in U$. Then, there is an open dense subset O_1' of $O_1 \cap \pi_{Q,M}^{-1}(U)$ such that, for every $q = (x, \hat{x}; A) \in O_1'$, there exists an open neighbourhood $\hat{V}$ of $\hat{x}$ where either $\hat{K}_2 = \hat{K}$ on $\hat{V}$ or $\hat{K}_2(\hat{y}) \neq \hat{K}(\hat{y})$ for $\hat{y} \in \hat{V}$. For the rest of the argument, we assume that q_1 belongs to O_1'. By shrinking U around x_1 and taking a small enough neighbourhood $\hat{U}$ of $\hat{x}_1$, we assume there are oriented orthonormal frames E_1, E_2, E_3 on U (resp. $\hat{E}_1, \hat{E}_2, \hat{E}_3$ on $\hat{U}$) such that $\star E_1, \star E_2, \star E_3$ (resp. $\hat{E}_1, \hat{E}_2, \hat{E}_3$) are eigenvectors of R (resp. $\hat{R}$) with eigenvalues $-K(\cdot), -K_2(\cdot), -K(\cdot)$ (resp. $-\hat{K}(\cdot), -\hat{K}_2(\cdot), -\hat{K}(\cdot)$), where these eigenvalues correspond to those in Proposition 5.22. Taking U, $\hat{U}$ smaller if necessary, we take X_A, Y_A, Z_A as given by Proposition 5.22 for M and $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$ for $\hat{M}$ on $\pi_Q^{-1}(U \times \hat{U}) \cap O_1'$, which we still denote by O_1'. Since $\star Y_A$ and $\star E_2|_x$ are both eigenvalues of $R|_x$, for $q = (x, \hat{x}; A) \in O_1'$, corresponding to single eigenvalue $-K_2(x)$, we moreover assume that $Y_A = E_2|_x$, $\forall q = (x, \hat{x}; A) \in O_1'$. Then because $\nu(\mathsf{Rol}_q(Z_A \wedge X_A))|_q = -K_2^{\mathsf{Rol}}(q)\nu(A \star E_2)|_q$ is tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ at the points $q = (x, \hat{x}; A) \in O_1'$, we conclude from Proposition 5.3 that

$$\Gamma = \begin{pmatrix} \Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)} \end{pmatrix},$$

where Γ and $\Gamma^i_{(j,k)}$ are as defined there.

We will now divide the proof in two parts (cases I and II below), depending whether $(\hat{M}, \hat{g})$ has, in certain areas, constant curvature or not.

Case I: Suppose, after shrinking $\hat{U}$ around x_1, that $\hat{K}_2(\hat{x}) = \hat{K}(\hat{x})$ for all $\hat{x} \in \hat{U}$. We also assume that $\hat{U}$ is connected. This implies by Schur Lemma (see [30], Proposition II.3.6) that $\hat{K}_2 = \hat{K}$ is constant on $\hat{U}$ and we write simply $\hat{K}$ for this constant. Again by possibly shrinking $\hat{U}$, we assume that $(\hat{U}, \hat{g}|_{\hat{U}})$ is isometric to an open subset of a 3-sphere of curvature $\hat{K}$.

Assume first that $\Gamma^1_{(2,3)} \neq 0$ on U. Then Proposition C.17, case (ii), implies that $\Gamma^1_{(1,2)} = 0$ on U and $(\Gamma^1_{(2,3)})^2 = K(x)$ is constant on U, which must be $\hat{K}$. Hence if $\beta := \Gamma^1_{(2,3)}$, which is constant on U, then $(U, g|_U)$ is of class $\mathcal{M}_\beta$ as is $(\hat{U}, \hat{g}|_{\hat{U}})$ and we are done (recall that $\mathcal{M}_{-\beta} = \mathcal{M}_\beta$) i.e., this is case (i). On the other hand, if $\Gamma^1_{(2,3)} = 0$ on U, then we have that $(U, g|_U)$, after possibly shrinking U, is isometric, by some F, to a warped product $(I \times N, h_f)$ by Proposition C.17 case (iii). At the same time, the space of constant curvature $(\hat{U}, \hat{g}|_{\hat{U}})$, again after shrinking $\hat{U}$ if necessary, can be presented, isometrically by certain $\hat{F}$, as a warped product $(\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$, where $\hat{N}$ is a 2-dimensional space of constant curvature. Because for all $x \in U$ we have $K(x) = \hat{K}$, we get that for all $(r, y) \in I \times N$, $\hat{r} \in \hat{I}$,

$$-\frac{f''(r)}{f(r)} = K(F(r, y)) = \hat{K} = -\frac{\hat{f}''(\hat{r})}{\hat{f}(\hat{r})}.$$

It is not hard to see that we may choose $\hat{f}$ such that $\hat{f}(0) = f(0)$ and $\hat{f}'(0) = f'(0)$, which then implies that $\hat{f}(r) = f(r)$, for all $r \in I$. This leads us to case (ii)

Case II: We assume here that $\hat{K}_2(\hat{x}) \neq \hat{K}(\hat{x})$ for all $\hat{x} \in \hat{U}$. The same way as for (M, g) above, this implies that $\hat{Y}_A = \hat{E}_2|_{\hat{x}}$ and that w.r.t. the frame $\hat{E}_1, \hat{E}_2, \hat{E}_3$, Proposition 5.3 yields

$$\hat{\Gamma} = \begin{pmatrix} \hat{\Gamma}^1_{(2,3)} & 0 & -\hat{\Gamma}^1_{(1,2)} \\ \hat{\Gamma}^1_{(3,1)} & \hat{\Gamma}^2_{(3,1)} & \hat{\Gamma}^3_{(3,1)} \\ \hat{\Gamma}^1_{(1,2)} & 0 & \hat{\Gamma}^1_{(2,3)} \end{pmatrix},$$

where $\hat{\Gamma}^i_{(j,k)} = \hat{g}(\hat{\nabla}_{\hat{E}_i}\hat{E}_j, \hat{E}_k)$, $1 \leq i,j,k \leq 3$.

We now claim that for all $(x,\hat{x}) \in \pi_Q(O_1')$, we have

$$\Gamma^1_{(2,3)}(x) = \hat{\Gamma}^1_{(2,3)}(\hat{x})$$
$$\Gamma^1_{(1,2)}(x) = \hat{\Gamma}^1_{(1,2)}(\hat{x}).$$

By Remark 5.23, we have $AY_A = \pm\hat{Y}_A$ for $q = (x,\hat{x};A) \in O_1'$, and so we get $AE_2|_x = \pm\hat{E}_2|_{\hat{x}}$. Without loss of generality, we assume that the '+' -case holds here. In particular, if $X \in \mathrm{VF}(M)$, one may differentiate the identity $AE_2 = \hat{E}_2$ w.r.t. $\mathscr{L}_{\mathrm{R}}(X)|_q$ to obtain

$$A\nabla_X E_2 = \hat{\nabla}_{AX}\hat{E}_2, \quad \forall q = (x,\hat{x};A) \in O_1'.$$

Since $AE_1, AE_2, \hat{E}_1, \hat{E}_2 \in (AE_2)^\perp = \hat{E}_2^\perp$, there exists, for every $q \in O_1'$, $\varphi = \varphi(q) \in \mathbb{R}$ such that

$$AE_1|_x = \cos(\varphi(q))\hat{E}_1|_{\hat{x}} + \sin(\varphi(q))\hat{E}_3|_{\hat{x}}$$
$$AE_3|_x = -\sin(\varphi(q))\hat{E}_1|_{\hat{x}} + \cos(\varphi(q))\hat{E}_3|_{\hat{x}}.$$

As usual, we write below $\cos(\varphi(q)) = c_\varphi$, $\sin(\varphi(q)) = s_\varphi$ Having this, we compute

$$\begin{aligned} A\nabla_{E_1}E_2 =& A(-\Gamma^1_{(1,2)}E_1 + \Gamma^1_{(2,3)}E_3) \\ =& (-c_\varphi\Gamma^1_{(1,2)} - s_\varphi\Gamma^1_{(2,3)})\hat{E}_1 + (-s_\varphi\Gamma^1_{(1,2)} + c_\varphi\Gamma^1_{(2,3)})\hat{E}_3, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \hat{\nabla}_{AE_1}\hat{E}_2 =& c_\varphi(-\hat{\Gamma}^1_{(1,2)}\hat{E}_1 + \hat{\Gamma}^1_{(2,3)}\hat{E}_3) + s_\varphi(-\hat{\Gamma}^1_{(2,3)}\hat{E}_1 - \hat{\Gamma}^1_{(1,2)}\hat{E}_3) \\ =& (-c_\varphi\hat{\Gamma}^1_{(1,2)} - s_\varphi\hat{\Gamma}^1_{(2,3)})\hat{E}_1 + (c_\varphi\hat{\Gamma}^1_{(2,3)} - s_\varphi\hat{\Gamma}^1_{(1,2)})\hat{E}_3. \end{aligned}$$

Taking $X = E_1$ above and using the last two formulas, we get

$$\begin{aligned} &(-c_\varphi\Gamma^1_{(1,2)} - s_\varphi\Gamma^1_{(2,3)})\hat{E}_1 + (-s_\varphi\Gamma^1_{(1,2)} + c_\varphi\Gamma^1_{(2,3)})\hat{E}_3 = A\nabla_{E_1}E_2 \\ =&\hat{\nabla}_{AE_1}\hat{E}_2 = (-c_\varphi\hat{\Gamma}^1_{(1,2)} - s_\varphi\hat{\Gamma}^1_{(2,3)})\hat{E}_1 + (c_\varphi\hat{\Gamma}^1_{(2,3)} - s_\varphi\hat{\Gamma}^1_{(1,2)})\hat{E}_3, \end{aligned}$$

from which

$$c_\varphi(-\Gamma^1_{(1,2)} + \hat{\Gamma}^1_{(1,2)}) + s_\varphi(\Gamma^1_{(2,3)} - \hat{\Gamma}^1_{(2,3)}) = 0.$$

Next we notice that differentiating the identity $AE_1 = c_\varphi\hat{E}_1 + s_\varphi\hat{E}_3$ w. r. t. $\nu(A \star E_2)|_q$ gives

$$A(\star E_2)E_1 = (\nu(A \star E_2)|_q\varphi)(-s_\varphi\hat{E}_1 + c_\varphi\hat{E}_3),$$

which simplifies to

$$-AE_3 = (\nu(A \star E_2)|_q\varphi)AE_3,$$

and hence yields

$$\nu(A \star E_2)|_q\varphi = -1, \quad \forall q = (x,\hat{x};A) \in O_1'.$$

Thus, if $(t,q)\mapsto\Phi(t,q)$ is the flow of $\nu((\cdot)\star E_2)$ in O_2' with initial position at $t=0$ at $q\in O_1'$, the above implies that $\varphi(\Phi(t,q))=\varphi(q)+t$ for all t such that $|t|$ is small enough. Since sin and cos are linearly independent functions on any non-empty open real interval, the above relation implies that

$$\begin{aligned}-\Gamma^1_{(1,2)}(x)+\hat{\Gamma}^1_{(1,2)}(\hat{x})&=0,\\ \Gamma^1_{(2,3)}(x)-\hat{\Gamma}^1_{(2,3)}(\hat{x})&=0,\end{aligned}$$

which establishes the claim.

We may now finish the proof of the proposition. Indeed, if $\Gamma^1_{(2,3)}\neq 0$ on U, Proposition C.17 implies that $\Gamma^1_{(2,3)}=:\beta$ is constant and $\Gamma^1_{(1,2)}=0$ on U. If $\hat{x}$ belongs to the open subset $\pi_{Q,\hat{M}}(O_1')$ of $\hat{M}$, there is a $q=(x,\hat{x};A)\in O_1'$ where $(x,\hat{x})\in U\times\hat{U}$, by the definition of O_1'. The above implies

$$\hat{\Gamma}^1_{(1,2)}(\hat{x})=\Gamma^1_{(1,2)}(x)=0,\quad \hat{\Gamma}^1_{(2,3)}(\hat{x})=\Gamma^1_{(2,3)}(x)=\beta.$$

Thus shrinking $\hat{U}$ if necessary, this shows that $\hat{\Gamma}^1_{(1,2)}$ vanishes on $\hat{U}$ and $\hat{\Gamma}^1_{(2,3)}$ is constant $=\beta$ on $\hat{U}$. We conclude that $(U,g|_U)$ and $(\hat{U},\hat{g}|_{\hat{U}})$ both belong to the class $\mathcal{M}_\beta$ and we are in case (i).

Similarly, if $\Gamma^1_{(2,3)}=0$ on U, the above argument implies that, after taking smaller $\hat{U}$, that $\hat{\Gamma}^1_{(2,3)}=0$ on $\hat{U}$. Proposition C.17 implies that there is, taking smaller $U,\hat{U}$ if needed, open interval $I=\hat{I}\subset\mathbb{R}$, smooth functions $f,\hat{f}:I=\hat{I}\to\mathbb{R}$, 2-dimensional Riemannian manifolds (N,h), $(\hat{N},\hat{h})$ and isometries $F:(I\times N,h_f)\to(U,g|_U)$, $\hat{F}:(\hat{I}\times\hat{N},\hat{h}_{\hat{f}})\to\hat{U}$ such that

$$\begin{aligned}\frac{f'(r)}{f(r)}&=\Gamma^1_{(1,2)}(F(r,y)),\quad\forall(r,y)\in I\times N,\\ \frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})}&=\hat{\Gamma}^1_{(1,2)}(\hat{F}(\hat{r},\hat{y})),\quad\forall(\hat{r},\hat{y})\in\hat{I}\times\hat{N}.\end{aligned}$$

Clearly we may assume that $0\in I=\hat{I}$ and $F(0,y_1)=x_1$, $\hat{F}(0,\hat{y}_1)=\hat{x}_1$ for some $y_1\in N$, $\hat{y}_1\in\hat{N}$. Since $t\mapsto(t,y_1)$ and $t\mapsto(t,\hat{y}_1)$ are geodesics in $(I\times N,h_f)$, $(\hat{I}\times\hat{N},\hat{h}_{\hat{f}})$, respectively, $\gamma(t):=F(t,y_1)$ and $\hat{\gamma}(t)=\hat{F}(t,\hat{y}_1)$ are geodesics on M and $\hat{M}$. In addition,

$$\hat{\gamma}'(0)=\hat{E}_2|_{\hat{x}_1}=A_1E_2|_{x_1}=A_1\gamma'(0),$$

so $\hat{\gamma}(t)=\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_1)(t)$ for all t. This means that

$$(F(t,y_1),\hat{F}(t,\hat{y}_1))=(\gamma(t),\hat{\gamma}(t))\in\pi_Q(O_1'),$$

and therefore

$$\frac{f'(t)}{f(t)}=\Gamma^1_{(1,2)}(F(t,y_1))=\hat{\Gamma}^1_{(1,2)}(\hat{F}(t,\hat{y}_1))=\frac{\hat{f}'(t)}{\hat{f}(t)},$$

for all $t\in I=\hat{I}$. We then belong to case (ii) and the proof of the proposition is concluded. ☐

We have studied the case where q belongs to $O_1 \cup O_2$. As for the points of O_0, one uses Corollary 4.16 and Remark 4.17 to conclude that for every $q_0 = (x_0, \hat{x}_0; A_0) \in O_0$, there are open neighbourhoods $U \ni x_0$ and $\hat{U} \ni \hat{x}_0$ such that $(U, g|_U)$ and $(\hat{U}, \hat{g}|_{\hat{U}})$ are locally isometric. With the choice of the set O as the union of $O_0 \cup O_1^\circ \cup O_2$, (where O_1° was introduced in Proposition 5.28), one concludes the proof of Theorem 5.1.

5.3 Proof of Theorem 5.2

Only Items (b) and (c) are addressed and they are treated in separate subsections.

5.3.1 Case where both Manifolds are of Class $\mathcal{M}_\beta$

Consider two manifolds (M, g) and $(\hat{M}, \hat{g})$ of class $\mathcal{M}_\beta$, $\beta \geq 0$ and oriented orthonormal frames E_1, E_2, E_3 and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ which are adapted frames for of (M, g) and $(\hat{M}, \hat{g})$ respectively. We will prove that in this situation, the rolling problem is not completely controllable.

We define on Q two subsets

$$\begin{aligned} Q_0 :=& \{q = (x, \hat{x}; A) \in Q \mid AE_2 \neq \pm \hat{E}_2\}, \\ Q_1 :=& \{q = (x, \hat{x}; A) \in Q \mid AE_2 = \pm \hat{E}_2\}. \end{aligned}$$

Proposition 5.29 Let (M, g), $(\hat{M}, \hat{g})$ be of class $\mathcal{M}_\beta$ for $\beta \in \mathbb{R}$. Then for any $q_0 = (x_0, \hat{x}_0; A_0) \in Q_1$ one has $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset Q_1$. Moreover, Q_1 is a closed 7-dimensional submanifold of Q and hence in particular $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 7$.

Proof. Define $h_1, h_2 : Q \to \mathbb{R}$ by

$$h_1(q) = \hat{g}(AE_1, \hat{E}_2), \quad h_2(q) = \hat{g}(AE_3, \hat{E}_2),$$

when $q = (x, \hat{x}; A) \in Q$. Set $h = (h_1, h_2) : Q \to \mathbb{R}^2$, then $Q_1 = h^{-1}(0)$. We will first show that h is regular at the points of Q_1, which then implies that Q_1 is a closed submanifold of Q of codimension 2 i.e., $\dim Q_1 = 7$ as claimed. Before proceeding, we divide Q_1 into two disjoint subsets

$$\begin{aligned} Q_1^+ =& \{q = (x, \hat{x}; A) \in Q \mid AE_2 = +\hat{E}_2\}, \\ Q_1^- =& \{q = (x, \hat{x}; A) \in Q \mid AE_2 = -\hat{E}_2\}, \end{aligned}$$

whence $Q = Q_1^+ \cup Q_1^-$. These are the components of Q and we prove the claims only for Q_1^+, the considerations for Q_1^- being completely similar. First, since for every $q = (x, \hat{x}; A) \in Q_1^+$ one has $AE_2 = \hat{E}_2$, it follows that $AE_1, AE_3 \in \hat{E}_2^\perp$ and hence there is a smooth $\phi : Q_1^+ \to \mathbb{R}$ such that

$$\begin{aligned} AE_1 =& \cos(\phi)\hat{E}_1 + \sin(\phi)\hat{E}_3 =: \hat{X}_A, \\ AE_3 =& -\sin(\phi)\hat{E}_1 + \cos(\phi)\hat{E}_3 =: \hat{Z}_A. \end{aligned}$$

In the sequel, we set $c_\phi = \cos(\phi(q))$, $s_\phi = \sin(\phi(q))$. For $q = (x, \hat{x}; A) \in Q_1^+$, one has

$$\begin{aligned} \nu(A \star E_3)|_q h_1 &= \hat{g}(A(\star E_3)E_1, \hat{E}_2) = \hat{g}(AE_2, \hat{E}_2) = 1, \\ \nu(A \star E_1)|_q h_1 &= \hat{g}(A(\star E_1)E_1, \hat{E}_2) = 0, \\ \nu(A \star E_3)|_q h_2 &= \hat{g}(A(\star E_3)E_3, \hat{E}_2) = 0, \\ \nu(A \star E_1)|_q h_2 &= \hat{g}(A(\star E_1)E_3, \hat{E}_2) = -\hat{g}(AE_2, \hat{E}_2) = -1, \end{aligned}$$

which shows that indeed h is regular on Q_1^+. We next show that the vectors $\mathscr{L}_{\mathrm{R}}(E_1)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(E_3)|_q$ are all tangent to Q_1^+ and hence to Q_1. This is equivalent to the fact that $\mathscr{L}_{\mathrm{R}}(E_i)|_q h = 0$ for $i = 1, 2, 3$. We compute for $q = (x, \hat{x}; A) \in Q_1^+$, recalling that $AE_1 = \hat{X}_A$, $AE_2 = \pm\hat{E}_2$, $AE_3 = \hat{Z}_A$,

$$
\begin{aligned}
\mathscr{L}_{\mathrm{R}}(E_1)|_q h_1 =& \hat{g}(A\nabla_{E_1}E_1, \hat{E}_2) + \hat{g}(AE_1, \hat{\nabla}_{\hat{X}_A}\hat{E}_2) \\
=& -\Gamma^1_{(3,1)}\hat{g}(AE_3, \hat{E}_2) + \hat{g}(\hat{X}_A, \beta c_\phi \hat{E}_3 - \beta s_\phi \hat{E}_1) \\
=& -\Gamma^1_{(3,1)}\hat{g}(\hat{Z}_A, \hat{E}_2) + \hat{g}(\hat{X}_A, \beta\hat{Z}_A) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_1)|_q h_2 =& \hat{g}(A\nabla_{E_1}E_3, \hat{E}_2) + \hat{g}(AE_3, \hat{\nabla}_{\hat{X}_A}\hat{E}_2) \\
=& \hat{g}(A(\Gamma^1_{(3,1)}E_1 - \beta E_2), \hat{E}_2) + \hat{g}(\hat{Z}_A, \beta\hat{Z}_A) \\
=& \hat{g}(\Gamma^1_{(3,1)}\hat{X}_A - \beta\hat{E}_2, \hat{E}_2) + \beta = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q h_1 =& \hat{g}(A\nabla_{E_2}E_1, \hat{E}_2) + \hat{g}(AE_1, \hat{\nabla}_{\hat{E}_2}\hat{E}_2) = -\Gamma^2_{(3,1)}\hat{g}(\hat{Z}_A, \hat{E}_2) + 0 = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q h_2 =& \hat{g}(A\nabla_{E_2}E_3, \hat{E}_2) + \hat{g}(AE_3, \hat{\nabla}_{\hat{E}_2}\hat{E}_2) = \Gamma^2_{(3,1)}\hat{g}(\hat{X}_A, \hat{E}_2) + 0 = 0, \\
\mathscr{L}_{\mathrm{R}}(E_3)|_q h_1 =& \hat{g}(A\nabla_{E_3}E_1, \hat{E}_2) + \hat{g}(AE_1, \hat{\nabla}_{\hat{Z}_A}\hat{E}_2) \\
=& \hat{g}(A(\beta E_2 - \Gamma^3_{(3,1)}E_3), \hat{E}_2) + \hat{g}(\hat{X}_A, -\beta s_\phi\hat{E}_3 - \beta c_\phi \hat{E}_1) \\
=& \hat{g}(\beta\hat{E}_2 - \Gamma^3_{(3,1)}\hat{Z}_A), \hat{E}_2) - \beta\hat{g}(\hat{X}_A, \hat{X}_A) = \beta - \beta = 0, \\
\mathscr{L}_{\mathrm{R}}(E_3)|_q h_2 =& \hat{g}(A\nabla_{E_3}E_3, \hat{E}_2) + \hat{g}(AE_3, \hat{\nabla}_{\hat{Z}_A}\hat{E}_2) \\
=& \Gamma^3_{(3,1)}\hat{g}(AE_1, \hat{E}_2) + \hat{g}(\hat{Z}_A, -\beta\hat{X}_A) = \Gamma^3_{(3,1)}\hat{g}(\hat{X}_A, \hat{E}_2) + 0 = 0.
\end{aligned}
$$

Thus $\mathscr{L}_{\mathrm{R}}(E_1)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(E_3)|_q$ and hence $\mathcal{D}_{\mathrm{R}}$ are tangent to Q_1^+, which implies that any orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ through a point $q \in Q_1^+$, is also a subset of Q_1^+. The same observation obviously holds for Q_1^- and therefore the proof is complete. □

Next we will show that if (M, g) and $(\hat{M}, \hat{g})$ are of class $\mathcal{M}_\beta$ with the same $\beta \in \mathbb{R}$, then the rolling problem of M against $\hat{M}$ is not controllable. We begin by completing the proposition in the sense that we show that the orbit can be of dimension exactly 7, if (M, g), $(\hat{M}, \hat{g})$ are not locally isometric.

Proposition 5.30 Let (M, g), $(\hat{M}, \hat{g})$ be Riemannian manifolds of class $\mathcal{M}_\beta$, $\beta \neq 0$, and let $q_0 = (x_0, \hat{x}_0; A_0) \in Q_1$. Then if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, one has $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 7$.

Proof. Without loss of generality, we assume that $A_0E_2|_{x_0} = \hat{E}_2|_{\hat{x}_0}$. Then Proposition 5.29 and continuity imply that $AE_2|_x = \hat{E}_2|_{\hat{x}}$ for all $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and hence that $AE_1|_x, AE_3|_x \in \mathrm{span}\{\hat{E}_1|_{\hat{x}}, \hat{E}_3|_{\hat{x}}\}$. This combined with Lemma C.8 implies

$$
\widetilde{\mathsf{Rol}}_q(\star E_1) = 0, \quad \widetilde{\mathsf{Rol}}_q(\star E_2) = (-K_2(x) + \hat{K}_2(\hat{x}))(\star E_2), \quad \widetilde{\mathsf{Rol}}_q(\star E_3) = 0,
$$

for $q = (x, \hat{x}; A) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, where $-K_2(x), -\hat{K}_2(\hat{x})$ are eigenvalues of $R|_x, \hat{R}|_{\hat{x}}$ corresponding to eigenvectors $\star E_2|_x, \hat{\star}\hat{E}_2|_{\hat{x}}$, respectively. Since $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, there is a point $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $-K_2(x_1) + \hat{K}_2(\hat{x}_1) \neq 0$ (see Corollary 4.16 and Remark 4.17). Then there are open neighbourhoods U and $\hat{U}$ of x_1 and $\hat{x}_1$ in M and $\hat{M}$, respectively, such that $-K_2(x) + \hat{K}_2(\hat{x}) \neq 0$ for all $x \in U, \hat{x} \in \hat{U}$. Define $O := \pi_Q^{-1}(U \times \hat{U}) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, which is an open subset of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ containing q_0. Because for all $q=(x,\hat{x};A)\in O$ one has $\nu(\mathsf{Rol}_q(\star E_2))|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and $-K_2(x)+\hat{K}_2(\hat{x})\neq 0$, it follows that

$$\nu(A\star E_2)|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),\quad \forall q=(x,\hat{x};A)\in O.$$

Moreover, $\Gamma^1_{(1,2)}=0$ and $\Gamma^1_{(2,3)}=\beta$ is constant and hence one may use Proposition 5.5, case (i), to conclude that the vector fields defined by

$$\begin{aligned}L_1|_q=&\mathscr{L}_{\mathrm{NS}}(E_1)|_q-\beta\nu(A\star E_1)|_q,\\ L_2|_q=&\beta\mathscr{L}_{\mathrm{NS}}(E_2)|_q,\\ L_3|_q=&\mathscr{L}_{\mathrm{NS}}(E_3)|_q-\beta\nu(A\star E_2)|_q,\end{aligned}$$

are tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Therefore the linearly independent vectors

$$\mathscr{L}_{\mathrm{R}}(E_1)|_q,\mathscr{L}_{\mathrm{R}}(E_2)|_q,\mathscr{L}_{\mathrm{R}}(E_3)|_q,\nu(A\star E_2)|_q,L_1|_q,L_2|_q,L_3|_q,$$

are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $q\in O$, which implies that $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\geq 7$. By Proposition 5.29, we conclude that $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=7$. □

We are left to study the case of an $\mathcal{D}_{\mathrm{R}}$-orbit passing through some $q_0\in Q_0$.

Proposition 5.31 Let (M,g) and $(\hat{M},\hat{g})$ be two Riemannian manifolds of class $\mathcal{M}_\beta$, $\beta\neq 0$, and let $q_0=(x_0,\hat{x}_0;A_0)\in Q_0$. Write $M^\circ:=\pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $\hat{M}^\circ:=\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, which are open connected subsets of M, $\hat{M}$. Then we have:

(i) If only one of (M°,g) or $(\hat{M}^\circ,\hat{g})$ has constant curvature, then $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=7$.

(ii) Otherwise $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=8$.

Proof. As before, we let E_1,E_2,E_3 and $\hat{E}_1,\hat{E}_2,\hat{E}_3$ to be some adapted frames of (M,g) and $(\hat{M},\hat{g})$ respectively. We will not fix the choice of q_0 in Q_0 (and hence do not define M°, $\hat{M}^\circ$) until the last half of the proof (where we introduce the sets $M_0,M_1,\hat{M}_0,\hat{M}_1$ below). Notice that Proposition 5.29 implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset Q_0$, for every $q_0\in Q_0$.

The fact that $AE_2|_x\neq\pm\hat{E}_2|_{\hat{x}}$ for $q=(x,\hat{x};A)\in Q_0$ is equivalent to the fact that the intersection $(AE_2^\perp|_x)\cap\hat{E}_2^\perp|_{\hat{x}}$ is non-trivial for all $q=(x,\hat{x};A)\in Q_0$. Therefore, for a small enough open neighbourhood $\tilde{O}$ of q_0 inside Q_0, we may find a smooth functions $\theta,\hat{\theta}:\tilde{O}\to\mathbb{R}$ such that this intersection is spanned by $AZ_A=\hat{Z}_A$, where

$$\begin{aligned}Z_A:=&-\sin(\theta(q))E_1|_x+\cos(\theta(q))E_3|_x,\\ \hat{Z}_A:=&-\sin(\hat{\theta}(q))\hat{E}_1|_{\hat{x}}+\cos(\hat{\theta}(q))\hat{E}_3|_{\hat{x}}.\end{aligned}$$

We also define

$$\begin{aligned}X_A:=&\cos(\theta(q))E_1|_x+\sin(\theta(q))E_3|_x,\\ \hat{X}_A:=&\cos(\hat{\theta}(q))\hat{E}_1|_{\hat{x}}+\sin(\hat{\theta}(q))\hat{E}_3|_{\hat{x}}.\end{aligned}$$

To unburden the formulas, we set $s_\tau:=\sin(\tau(q))$, $c_\tau:=\cos(\tau(q))$ if $\tau:\tilde{O}\to\mathbb{R}$ is some function and the point $q\in\tilde{O}$ is clear from the context. Since $X_A,E_2|_x,Z_A$

(resp. $\hat{X}_A, \hat{E}_2|_{\hat{x}}, \hat{Z}_A$) form an orthonormal frame for every $q=(x,\hat{x};A)\in\tilde{O}$ and because $A(Z_A^{\perp})=\hat{Z}_A^{\perp}$, it follows that there is a smooth $\phi: O'\to\mathbb{R}$ such that

$$\begin{aligned}
AX_A =& c_{\hat{\phi}}\hat{X}_A + s_{\hat{\phi}}\hat{E}_2 = c_{\hat{\phi}}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + s_{\hat{\phi}}\hat{E}_2,\\
AE_2 =& - s_{\hat{\phi}}\hat{X}_A + c_{\hat{\phi}}\hat{E}_2 = -s_{\hat{\phi}}(c_{\hat{\theta}}\hat{E}_1 + s_{\hat{\theta}}\hat{E}_3) + c_{\hat{\phi}}\hat{E}_2,\\
AZ_A =& \hat{Z}_A.
\end{aligned}$$

In particular, for all $q=(x,\hat{x};A)\in\tilde{O}$, one has $\hat{g}(AZ_A,\hat{E}_2)=0$. Note that for all $q=(x,\hat{x};A)\in\tilde{O}$, since $A\star Z_A=\hat{\star}\hat{Z}_A A$,

$$\widetilde{\mathsf{Rol}}_q(\star Z_A) = R(\star Z_A) - A^{\overline{T}}\hat{R}(\hat{\star}\hat{Z}_A)A = -K\star Z_A + KA^{\overline{T}}\hat{\star}\hat{Z}_A A = 0,$$

and hence, since $\widetilde{\mathsf{Rol}}_q : \wedge^2 T|_x M\to\wedge^2 T|_x M$ is a symmetric map,

$$\begin{aligned}
\widetilde{\mathsf{Rol}}_q(\star X_A) =& - K_1^{\mathsf{Rol}}(q)\star X_A - \alpha\star E_2,\\
\widetilde{\mathsf{Rol}}_q(\star E_2) =& - \alpha\star X_A - K_2^{\mathsf{Rol}}(q)\star E_2,
\end{aligned}$$

for some smooth real-valued functions $K_1^{\mathsf{Rol}}, K_2^{\mathsf{Rol}}, \alpha$ defined on $\tilde{O}$.

We begin by considering the smooth 5-dimensional distribution Δ on the open subset $\tilde{O}$ of Q_0 spanned by

$$\mathscr{L}_{\mathrm{R}}(E_1)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(E_3)|_q, \nu(A\star E_2)|_q, \nu(A\star X_A)|_q.$$

What will be shown is that $\mathrm{Lie}(\Delta)$ spans at every point $q\in\mathcal{O}$ a smooth distribution $\mathrm{Lie}(\Delta)|_q$ of dimension 8 which, by construction, is involutive. We consider $\mathrm{VF}^k_{\mathcal{D}_{\mathrm{R}}}, \mathrm{VF}^k_{\Delta}, \mathrm{Lie}(\Delta)$ as $C^\infty(\tilde{O})$-modules. Since $X_A = c_\theta E_1 + s_\theta E_3$, in order to compute brackets of the first 4 vector fields above against $\nu(A\star X_A)|_q$, we need to know some derivatives of θ. We begin by computing the following.

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q Z_{(\cdot)} =& (-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})X_A - \beta E_2,\\
\mathscr{L}_{\mathrm{R}}(E_2)|_q Z_{(\cdot)} =& (-\mathscr{L}_{\mathrm{R}}(E_2)|_q\theta + \Gamma^2_{(3,1)})X_A,\\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q Z_{(\cdot)} =& (-\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})X_A.
\end{aligned}$$

Differentiating $\hat{g}(AZ_A,\hat{E}_2)=0$ with respect to $\mathscr{L}_{\mathrm{R}}(X_A)|_q$ gives,

$$\begin{aligned}
0 =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(X_A)|_q Z_{(\cdot)}, \hat{Y}) + \hat{g}(AZ_A, \hat{\nabla}_{AX_A}\hat{E}_2)\\
=& \hat{g}(A(-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})X_A - \beta E_2), \hat{E}_2)\\
&+ \hat{g}(AZ_A, c_{\hat{\phi}}c_{\hat{\theta}}\beta\hat{E}_3 - c_{\hat{\phi}}s_{\hat{\theta}}\beta\hat{E}_1)\\
=& s_{\hat{\phi}}(-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}) - \beta c_{\hat{\phi}} + c_{\hat{\phi}}s^2_{\hat{\theta}}\beta + c_{\hat{\phi}}c^2_{\hat{\theta}}\beta\\
=& s_{\hat{\phi}}(-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}).
\end{aligned}$$

Since $s_{\hat{\phi}}\neq 0$ (because otherwise $AE_2=\pm\hat{E}_2$), we get

$$\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta = c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}.$$

In a similar way, differentiating $\hat{g}(AZ_A,\hat{E}_2)=0$ with respect to $\mathscr{L}_{\mathrm{R}}(Z_A)|_q$, $\mathscr{L}_{\mathrm{R}}(E_2)|_q$, one finds

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta =& -s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)},\\
\mathscr{L}_{\mathrm{R}}(E_2)|_q\theta =& -\beta + \Gamma^2_{(3,1)}.
\end{aligned}$$

Finally, applying $\nu(A\star E_2)|_q$ on the equation $\hat{g}(AZ_A,\hat{E}_2)=0$ gives,

$$\begin{aligned}0=&\hat{g}(\nu(A\star E_2)|_q((\cdot)Z_{(\cdot)},\hat{E}_2)=\hat{g}(A(\star E_2)Z_A-(\nu(A\star E_2)|_q\theta)AX_A,\hat{E}_2)\\ =&(1-\nu(A\star E_2)|_q\theta)\hat{g}(AX_A,\hat{E}_2),\end{aligned}$$

and since $\hat{g}(AX_A,\hat{E}_2)=s_{\hat{\phi}}\neq 0$, $\nu(A\star E_2)|_q\theta=1$. Using the definition of X_A and Z_A, we may now summarize

$$\begin{aligned}&\mathscr{L}_{\mathrm{R}}(E_1)|_q\theta=\Gamma^1_{(3,1)}, \quad \mathscr{L}_{\mathrm{R}}(E_2)|_q\theta=-\beta+\Gamma^2_{(3,1)},\\ &\mathscr{L}_{\mathrm{R}}(E_3)|_q\theta=\Gamma^3_{(3,1)}, \quad \nu(A\star E_2)|_q\theta=1.\end{aligned}$$

By Proposition 5.5 and the fact that $\beta\neq 0$, we see that VF^2_Δ contains the vector fields given by

$$\begin{aligned}L_1|_q=&\mathscr{L}_{\mathrm{NS}}(E_1)|_q-\beta\nu(A\star E_1)|_q,\\ \tilde{L}_2|_q=&\mathscr{L}_{\mathrm{NS}}(E_2)|_q,\\ L_3|_q=&\mathscr{L}_{\mathrm{NS}}(E_3)|_q-\beta\nu(A\star E_3)|_q,\end{aligned}$$

i.e., $\tilde{L}_2=\frac{1}{\beta}L_2$. Computing

$$\begin{aligned}[\mathscr{L}_{\mathrm{R}}(E_1),\nu((\cdot)\star X_{(\cdot)})]|_q=&-s_\theta\mathscr{L}_{\mathrm{R}}(E_2)|_q+s_\theta\tilde{L}_2|_q-s_\theta\beta\nu(A\star E_2)|_q,\\ [\mathscr{L}_{\mathrm{R}}(E_2),\nu((\cdot)\star X_{(\cdot)})]|_q=&-\mathscr{L}_{\mathrm{R}}(Z_A)|_q-s_\theta L_1|_q+c_\theta L_3|_q,\\ [\mathscr{L}_{\mathrm{R}}(E_3),\nu((\cdot)\star X_{(\cdot)})]|_q=&c_\theta\mathscr{L}_{\mathrm{R}}(E_2)|_q-c_\theta\tilde{L}_2|_q-c_\theta\beta\nu(A\star E_2)|_q,\\ [\nu((\cdot)\star E_2),\nu((\cdot)\star X_{(\cdot)})]|_q,=&0\end{aligned}$$

and since one also has

$$\begin{aligned}[\mathscr{L}_{\mathrm{R}}(E_1),\mathscr{L}_{\mathrm{R}}(E_2)]|_q=&\mathscr{L}_{\mathrm{R}}([E_1,E_2])|_q-s_\theta K_1^{\mathsf{Rol}}\nu(A\star X_A)|_q-s_\theta\alpha\nu(A\star E_2)|_q,\\ [\mathscr{L}_{\mathrm{R}}(E_2),\mathscr{L}_{\mathrm{R}}(E_3)]|_q=&\mathscr{L}_{\mathrm{R}}([E_2,E_3])|_q-c_\theta K_1^{\mathsf{Rol}}\nu(A\star X_A)|_q-c_\theta\alpha\nu(A\star E_2)|_q,\\ [\mathscr{L}_{\mathrm{R}}(E_3),\mathscr{L}_{\mathrm{R}}(E_1)]|_q=&\mathscr{L}_{\mathrm{R}}([E_3,E_1])|_q-\alpha\nu(A\star X_A)|_q-K_2^{\mathsf{Rol}}\nu(A\star E_2)|_q,\end{aligned}$$

we see using in addition Proposition 5.5, case (ii) (the first three Lie brackets there), that VF^2_Δ is generated by the following 8 linearly independent vector fields defined on $\tilde{O}$ by

$$\mathscr{L}_{\mathrm{R}}(E_1)|_q,\mathscr{L}_{\mathrm{R}}(E_2)|_q,\mathscr{L}_{\mathrm{R}}(E_3)|_q,\nu(A\star E_2)|_q,\nu(A\star X_A)|_q,L_1|_q,\tilde{L}_2|_q,L_3|_q.$$

We now proceed to show that $\mathrm{Lie}(\Delta)=\mathrm{VF}^2_\Delta$. According to Proposition 5.5 case (ii) and the previous computations, we know that all the brackets between $\mathscr{L}_{\mathrm{R}}(E_1)$, $\mathscr{L}_{\mathrm{R}}(E_2)$, $\mathscr{L}_{\mathrm{R}}(E_3)$, $\nu((\cdot)\star E_2)$ and L_1,L_3 and also $[L_1,L_3]$ belong to VF^2_Δ, so we are left to compute the bracket of $\nu((\cdot)\star X_{(\cdot)}),\tilde{L}_2$ against L_1,L_3 and also $\tilde{L}_2$ against $\mathscr{L}_{\mathrm{R}}(E_1)|_q,\mathscr{L}_{\mathrm{R}}(E_2)|_q,\mathscr{L}_{\mathrm{R}}(E_3)|_q,\nu(A\star E_2)|_q,\nu((\cdot)\star X_{(\cdot)})|_q$. To do that, we need to know more derivatives of θ. Since $[\mathscr{L}_{\mathrm{R}}(E_1),\nu((\cdot)\star E_2)]=\mathscr{L}_{\mathrm{R}}(E_3)|_q-L_3|_q$, we get

$$L_3|_q\theta=\mathscr{L}_{\mathrm{R}}(E_3)|_q\theta-\mathscr{L}_{\mathrm{R}}(E_1)|_q\big(\underbrace{\nu((\cdot)\star E_2)\theta}_{=1}\big)+\nu(A\star E_2)|_q\big(\underbrace{\mathscr{L}_{\mathrm{R}}(E_1)\theta}_{=\Gamma^1_{(3,1)}}\big)=\Gamma^3_{(3,1)},$$

and similarly, by using $[\mathscr{L}_{\rm R}(E_3),\nu((\cdot)\star E_2)]=-\mathscr{L}_{\rm R}(E_1)|_q+L_1|_q$, one gets $L_1|_q\theta=\Gamma^1_{(3,1)}$. On the other hand, $\mathscr{L}_{\rm NS}(E_2)|_q Z_{(\cdot)}=(-\mathscr{L}_{\rm NS}(E_2)|_q\theta+\Gamma^2_{(3,1)})X_A$, and to compute $\tilde{L}_2|_q\theta=\mathscr{L}_{\rm NS}(E_2)|_q\theta$, operate by $\mathscr{L}_{\rm NS}(E_2)|_q$ onto equation $\hat{g}(AZ_A,\hat{E}_2)=0$ to get $\tilde{L}_2|_q\theta=\Gamma^2_{(3,1)}$. With these derivatives of θ being available, we easily see that

$$\begin{aligned}
[L_1,\nu((\cdot)\star X_{(\cdot)})]|_q=&0,\\
[L_1,\tilde{L}_2]|_q=&(\Gamma^2_{(3,1)}+\beta)L_3|_q,\\
[L_3,\nu((\cdot)\star X_{(\cdot)})]|_q=&0,\\
[L_3,\tilde{L}_2]|_q=&-(\Gamma^2_{(3,1)}+\beta)L_1|_q,\\
[\mathscr{L}_{\rm R}(E_1),\tilde{L}_2]|_q=&\beta L_3|_q-\mathscr{L}_{\rm R}(\nabla_{E_2}E_1)|_q,\\
[\mathscr{L}_{\rm R}(E_2),\tilde{L}_2]|_q=&0,\\
[\mathscr{L}_{\rm R}(E_3),\tilde{L}_2]|_q=&-\beta L_1|_q-\mathscr{L}_{\rm R}(\nabla_{E_2}E_3)|_q,\\
[\nu((\cdot)\star E_2),\tilde{L}_2]|_q=&0,\\
[\nu((\cdot)\star X_{(\cdot)}),\tilde{L}_2]|_q=&0.
\end{aligned}$$

Hence we have proved that VF^2_Δ is involutive and hence

$$\mathrm{Lie}(\Delta)=\mathrm{VF}^2_\Delta.$$

There being 8 linearly independent generators for $\mathrm{Lie}(\Delta)=\mathrm{VF}^2_\Delta$, we conclude that the distribution $\mathcal{D}$ spanned pointwise on $\tilde{O}$ by $\mathrm{Lie}(\Delta)$ is integrable by Frobenius theorem. The choice of $q_0\in Q_0$ was arbitrary and we thus can build an 8-dimensional smooth involutive distribution $\mathcal{D}$ by the above construction on the whole Q_0. Since $\mathcal{D}_{\rm R}\subset\Delta\subset\mathcal{D}$, we have $\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\subset\mathcal{O}_{\mathcal{D}}(q_0)$ for all $q_0\in Q_0$ and thus $\dim\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\le 8$.

We will show when the equality holds here and show when actually $\dim\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)=7$. Define

$$\begin{aligned}
M_0=&\{x\in M\mid \beta^2\neq K_2(x)\},\\
M_1=&\{x\in M\mid \exists\text{ open } V\ni x \text{ s.t. } \forall x'\in V,\ \beta^2=K_2(x')\},\\
\hat{M}_0=&\{\hat{x}\in \hat{M}\mid \beta^2\neq \hat{K}_2(\hat{x})\},\\
\hat{M}_1=&\{\hat{x}\in \hat{M}\mid \exists\text{ open } \hat{V}\ni \hat{x} \text{ s.t. } \forall \hat{x}'\in \hat{V},\ \beta^2=\hat{K}_2(\hat{x}')\},
\end{aligned}$$

and notice that $M_0\cup M_1$ (resp. $\hat{M}_0\cup\hat{M}_1$) is an open dense subset of M (resp. $\hat{M}$). At this point we also fix $q_0\in Q_0$ and write $M^\circ=\pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$, $\hat{M}^\circ=\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0))$ as in the statement of this proposition. Let $q_1=(x_1,\hat{x}_1;A_1)\in\pi_Q^{-1}(M_0\times\hat{M}_0)\cap Q_0$. Take an open neighbourhood $\tilde{O}$ of q_1 in Q_0 as above (now for q_1 instead of q_0 which we fixed) such that $\pi_Q(\tilde{O})\subset M_0\times\hat{M}_0$, and introduce on $\tilde{O}$ the vectors $X_A,Z_A,\hat{X}_A,\hat{Z}_A$ along with the angles $\theta,\hat{\theta},\hat{\phi}$, again as above. For $q\in\tilde{O}$, one has

$$\begin{aligned}
\begin{pmatrix}\widetilde{\mathsf{Rol}}_q(\star X_A)\\ \widetilde{\mathsf{Rol}}_q(\star E_2)\end{pmatrix}=&\begin{pmatrix} s^2_{\hat{\phi}}(-\beta^2+\hat{K}_2) & c_{\hat{\phi}}s_{\hat{\phi}}(-\beta^2+\hat{K}_2)\\ (-\beta^2+\hat{K}_2)s_{\hat{\phi}}c_{\hat{\phi}} & -K_2+s^2_{\hat{\phi}}\beta^2+c^2_{\hat{\phi}}\hat{K}_2\end{pmatrix}\begin{pmatrix}\star X_A\\ \star E_2\end{pmatrix},\\
\widetilde{\mathsf{Rol}}_q(\star Z_A)=&0.
\end{aligned}$$

The determinant $d(q)$ of the above matrix is equal to

$$d(q)=-s^2_{\hat{\phi}}(-K_2+\beta^2)(-\hat{K}_2+\beta^2),$$

so $d(q) \neq 0$ since $q \in \tilde{O} \subset \pi_Q^{-1}(M_0 \times \hat{M}_0) \cap Q_0$. Since $\nu(\mathsf{Rol}(\star E_2)(A))|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$, we obtain that $\nu(A_1 \star E_2)|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. If $q_1 = (x_1, \hat{x}_1; A_1) \in \overline{\pi_Q^{-1}(M_0 \times \hat{M}_0) \cap Q_0}$, then one can take a sequence $q'_n = (x'_n, \hat{x}'_n; A'_n) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$ such that $q'_n \to q_1$ while $\hat{x}'_n \in \hat{M}_0$. Since M_0 and Q_0 are open, we have for large enough n that $q'_n \in \pi_Q^{-1}(M_0 \times \hat{M}_0) \cap Q_0$, hence $\nu(A'_n \star E_2)|_{q'_n} \in T|_{q'_n}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$ and by taking the limit as $n \to \infty$, we have $\nu(A_1 \star E_2)|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Suppose next $q_1 = (x_1, \hat{x}_1; A_1) \in \pi_Q^{-1}(M_0 \times \hat{M}_1) \cap Q_0$. Then $\widetilde{\mathsf{Rol}}_{q_1}(\star E_1) = \widetilde{\mathsf{Rol}}_{q_1}(\star E_3) = 0$, $\widetilde{\mathsf{Rol}}_{q_1}(\star E_2) = (-K_2(x_1) + \beta^2) \star E_2$ with $K_2(x_1) \neq \beta^2$ and hence $\nu(A \star E_2)|_{q_1} \in T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Thus we have proven that

$$\nu(A \star E_2)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M_0 \times \hat{M}).$$

Changing the roles of M and $\hat{M}$ we also have

$$\nu((\hat{\star}\hat{E}_2)A)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M \times \hat{M}_0).$$

On Q, define two 3-dimensional distributions $D, \hat{D}$ as follows, for $q \in Q$ let $\hat{D}|_q$ be the span of

$$\begin{aligned}
\hat{K}_1|_q &= \mathscr{L}_{\mathrm{NS}}(AE_1)|_q + \beta\nu(A \star E_1)|_q, \\
\hat{K}_2|_q &= \mathscr{L}_{\mathrm{NS}}(AE_2)|_q, \\
\hat{K}_3|_q &= \mathscr{L}_{\mathrm{NS}}(AE_3)|_q + \beta\nu(A \star E_3)|_q,
\end{aligned}$$

and $D|_q$ be the span of

$$\begin{aligned}
K_1|_q &= \mathscr{L}_{\mathrm{NS}}(A^{\overline{T}}\hat{E}_1)|_q - \beta\nu((\hat{\star}\hat{E}_1)A)|_q, \\
K_2|_q &= \mathscr{L}_{\mathrm{NS}}(A^{\overline{T}}\hat{E}_2)|_q, \\
K_3|_q &= \mathscr{L}_{\mathrm{NS}}(A^{\overline{T}}\hat{E}_3)|_q - \beta\nu((\hat{\star}\hat{E}_3)A)|_q.
\end{aligned}$$

We claim that for any $q_1 = (x_1, \hat{x}_1; A_1) \in Q$ and any smooth paths $\gamma : [0,1] \to M$, $\hat{\gamma} : [0,1] \to \hat{M}$ with $\gamma(0) = x_1$, $\hat{\gamma}(0) = \hat{x}_1$ there are unique curves $\Gamma, \hat{\Gamma} : [0,1] \to Q$ of the same regularity as $\gamma, \hat{\gamma}$ such that Γ is tangent to D, $\Gamma(0) = q_1$ and $\pi_{Q,M}(\Gamma(t)) = \gamma$ and similarly $\hat{\Gamma}$ is tangent to $\hat{D}$, $\hat{\Gamma}(0) = q_1$ and $\pi_{Q,\hat{M}}(\hat{\Gamma}(t)) = \hat{\gamma}$. The key point here is that $\Gamma, \hat{\Gamma}$ are defined on $[0,1]$ and not only on a smaller interval $[0,T]$ with $T \leq 1$. We write these curves as $\Gamma = \Gamma(\gamma, q_1)$ and $\hat{\Gamma} = \hat{\Gamma}(\hat{\gamma}, q_1)$, respectively. Notice that since $(\pi_{Q,\hat{M}})_* D = 0$ and $(\pi_{Q,M})_* \hat{D} = 0$, one has

$$\pi_{Q,\hat{M}}(\Gamma(\gamma, q_1)(t)) = \hat{x}_1, \quad \pi_{Q,M}(\hat{\Gamma}(\hat{\gamma}, q_1)(t)) = x_1, \quad \forall t \in [0,1].$$

We only prove the above claim for D since the proof for $\hat{D}$ is similar. Uniqueness and local existence are straigthforward. Take some extension of γ to an interval $]-\epsilon, 1+\epsilon[=: I$ and write $\Gamma_1 := \Gamma(\gamma, q_1)$. Consider a trivialization (global since we assumed the frames E_i, $\hat{E}_i$, $i = 1,2,3$ to be global) of π_Q given by

$$\Phi : Q \to M \times \hat{M} \times \mathrm{SO}(n), \quad (x, \hat{x}; A) \mapsto (x, \hat{x}, \mathcal{M}_{F,\hat{F}}(A)),$$

where $F = (E_1, E_2, E_3)$, $\hat{F} = (\hat{E}_1, \hat{E}_2, \hat{E}_3)$. For every $(s, C) \in I \times \mathrm{SO}(n)$ one has

$$\Phi(\Gamma(\gamma(s + \cdot), \Phi^{-1}(\gamma(s), \hat{x}_1; C))(t)) = (\gamma(s + t), \hat{x}_1, B_{(s,C)}(t)),$$

where $B_{(s,C)}(t) \in \mathrm{SO}(n)$ and t in an open interval containing 0. On $I \times \mathrm{SO}(n)$, define a vector field

$$\mathcal{X}|_{(s,C)} := (\frac{\partial}{\partial t}, \dot{B}_{(s,C)}(0)).$$

If $\Phi(\Gamma(\gamma, q_1)(t)) = (\gamma(t), \hat{x}_1; C_1(t))$, then since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Phi(\Gamma_1(s)) =&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\Phi(\Gamma(\gamma,q_1)(t+s)) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0\Phi\big(\Gamma(\gamma(s+\cdot),\Gamma(\gamma,q_1)(s))(t)\big)\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0(\gamma(t+s),\hat{x}_1,B_{(s,C_1(s))}(t)) = (\dot{\gamma}(s),0,(\mathrm{pr}_2)_*\mathcal{X}|_{(s,C_1(s))}),\end{aligned}$$

we see that $s \mapsto (s, (\mathrm{pr}_3 \circ \Phi \circ \Gamma_1)(s)) = (s, C_1(s))$ is the integral curve of $\mathcal{X}$ starting from $(0, C_1(0))$. Conversely, if $\Lambda_1(t) = (t, C(t))$ is the integral curve of $\mathcal{X}$ starting from $(0, C_1(0))$, then $\tilde{\Gamma}_1(t) := \Phi^{-1}(\gamma(t), \hat{x}_1, C(t))$ gives an integral curve of D starting from q_1 and $\pi_{Q,M}(\tilde{\Gamma}_1(t)) = \gamma(t)$.

Hence the maximal positive interval of definition of Γ_1 is the same as that of the integral curve Λ_1 of $\mathcal{X}$ starting from $(0, C_1)$. If it is of the form $[0, t_0[$ for some $t_0 < 1 + \epsilon$, then, because $[0, 1] \times \mathrm{SO}(n)$ is a compact subset of $I \times \mathrm{SO}(n)$, there is a $t_1 \in [0, t_0[$ with $\Lambda_1(t_1) \notin [0, 1] \times \mathrm{SO}(n)$ i.e. $t_1 \notin [0, 1]$ which is only possible if $t_1 > 1$, and thus $t_0 > 1$. We have shown that the existence of $\Gamma_1(t) = \Gamma(\gamma, q_1)(t)$ is guaranteed on the whole interval $[0, 1]$.

Since for all $q \in Q_0 \cap \pi_Q^{-1}(M_0 \times \hat{M})$, which is an open subset of Q, one has $\nu(A \star E_2)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$, it follows from Proposition 5.5 that

$$\begin{aligned}L_1|_q &= \mathscr{L}_{\mathrm{NS}}(E_1)|_q - \beta\nu(A \star E_1)|_q,\\ \tilde{L}_2|_q &= \mathscr{L}_{\mathrm{NS}}(E_2)|_q,\\ L_3|_q &= \mathscr{L}_{\mathrm{NS}}(E_3)|_q - \beta\nu(A \star E_3)|_q,\end{aligned}$$

are tangent to the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ and hence so are $\mathscr{L}_{\mathrm{R}}(E_1)|_q - L_1|_q = \hat{K}_1|_q$, $\mathscr{L}_{\mathrm{R}}(E_2)|_q - \tilde{L}_2 = \hat{K}_2|_q$ and $\mathscr{L}_{\mathrm{R}}(E_3)|_q - L_3|_q = \hat{K}_3|_q$ i.e.,

$$\hat{D}|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M_0 \times \hat{M}).$$

A similar argument shows that

$$D|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M \times \hat{M}_0).$$

Assume now that $(M_1 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and that $M_0 \neq \emptyset$. Choose any $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ with $(x_1, \hat{x}_1) \in M_1 \times \hat{M}_0$ and take any curve $\gamma : [0, 1] \to M$ with $\gamma(0) = x_1$, $\gamma(1) \in M_0$. Then since $\pi_{Q,\hat{M}}(\Gamma(\gamma, q_1)(t)) = \hat{x}_1$, we have $\pi_Q(\Gamma(\gamma, q_1)(t)) \in M \times \hat{M}_0$ for all $t \in [0, 1]$ and since also $D|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap \pi_Q^{-1}(M \times \hat{M}_0)$, we have that $\Gamma(\gamma, q_1)(t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $t \in [0, 1]$. Indeed, suppose there is a $0 \leq t < 1$ with $\Gamma(\gamma, q_1)(t) \notin \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and define $t_1 = \inf\{t \in [0, 1] \mid \Gamma(\gamma, q_1)(t) \notin \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\}$. Clearly $t_1 > 0$. Because $q_2 := \Gamma(\gamma, q_1)(t_1) \in \pi_Q^{-1}(M \times \hat{M}_0)$, it follows that for $|t|$ small one has $\Gamma(\gamma, q_1)(t_1+t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_2)$, whence if $t < 0$ small, $\Gamma(\gamma, q_1)(t_1+t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_2) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, which means that $q_2 \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and thus for $t \geq 0$ small $\Gamma(\gamma, q_1)(t_1+t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, a contradiction. Hence one has $\pi_Q(\Gamma(\gamma, q_1)(1)) \in (M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. In other words we have the implication:

$$(M_1 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset, \quad M_0 \neq \emptyset \implies (M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset.$$

By a similar argument, using $\hat{D}$ instead of D, one has that

$$(M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset, \quad \hat{M}_0 \neq \emptyset \implies (M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset.$$

Suppose now that there exists $q_1 = (x_1, \hat{x}_1; A_1) \in \pi_Q^{-1}(M_0 \times \hat{M}_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. We already know that $T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ contains vectors

$$\begin{aligned}
&\mathscr{L}_{\mathrm{R}}(E_1)|_{q_1}, \mathscr{L}_{\mathrm{R}}(E_2)|_{q_1}, \mathscr{L}_{\mathrm{R}}(E_3)|_{q_1},\\
&\nu(A \star E_2)|_{q_1}, \nu((\hat{\star}\hat{E}_2)A)|_{q_1},\\
&L_1|_{q_1}, \tilde{L}_2|_{q_1}, L_3|_{q_1},
\end{aligned}$$

which are linearly independent since $q_1 \in (M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Indeed, if one introduces X_A, Z_A and an angle ϕ as before, we have $\sin(\phi(q_1)) \neq 0$ as $q_1 \in Q_0$ and

$$\nu((\hat{\star}\hat{E}_2)A_1)|_{q_1} = \nu(A_1 \star (A_1^T \hat{E}_2))|_{q_1} = \sin(\phi(q_1))\nu(A_1 \star X_{A_1})|_{q_1} + \cos(\phi(q_1))\nu(A_1 \star E_2)|_{q_1}.$$

Therefore $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 8$ and since we have also shown that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 8$, we have that

$$(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset \implies \dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$$

Write $Q^\circ := \pi_Q^{-1}(M^\circ \times \hat{M}^\circ)$, which is an open subset of Q and clearly $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset Q^\circ$. To finish the proof, we proceed case by case.

a) Suppose $(\hat{M}^\circ, \hat{g})$ has constant curvature i.e. $\hat{M}_0 \cap \hat{M}^\circ = \emptyset$. By assumption then, (M°, g) does not have constant curvature, which means that $M_0 \cap M^\circ \neq \emptyset$.

   At every $q = (x, \hat{x}; A) \in Q^\circ$, one has $\widetilde{\mathsf{Rol}}_q(\star E_1) = \widetilde{\mathsf{Rol}}_q(\star E_3) = 0$ and $\widetilde{\mathsf{Rol}}_q(\star E_2) = (-K_2(x) + \beta^2) \star E_2$ and therefore

   $$\begin{aligned}
   &[\mathscr{L}_{\mathrm{R}}(E_1), \mathscr{L}_{\mathrm{R}}(E_2)]|_q = \mathscr{L}_{\mathrm{R}}([E_1, E_2])|_q, \quad [\mathscr{L}_{\mathrm{R}}(E_2), \mathscr{L}_{\mathrm{R}}(E_3)]|_q = \mathscr{L}_{\mathrm{R}}([E_2, E_3])|_q,\\
   &[\mathscr{L}_{\mathrm{R}}(E_3), \mathscr{L}_{\mathrm{R}}(E_1)]|_q = \mathscr{L}_{\mathrm{R}}([E_3, E_1])|_q + (-K_2(x) + \beta^2)\nu(A \star E_2)|_q.
   \end{aligned}$$

   From these, Proposition 5.5 case (ii) and from the brackets (as above)

   $$\begin{aligned}
   [\mathscr{L}_{\mathrm{R}}(E_1), \tilde{L}_2]|_q =& \beta L_3|_q - \mathscr{L}_{\mathrm{R}}(\nabla_{E_2} E_1)|_q,\\
   [\mathscr{L}_{\mathrm{R}}(E_3), \tilde{L}_2]|_q =& -\beta L_1|_q - \mathscr{L}_{\mathrm{R}}(\nabla_{E_2} E_3)|_q,\\
   [\mathscr{L}_{\mathrm{R}}(E_2), \tilde{L}_2]|_q =& 0,\\
   [\nu((\cdot) \star E_2), \tilde{L}_2]|_q =& 0,\\
   [L_1, \tilde{L}_2]|_q =& (\Gamma^2_{(3,1)} + \beta)L_3|_q,\\
   [L_3, \tilde{L}_2]|_q =& -(\Gamma^2_{(3,1)} + \beta)L_1|_q,
   \end{aligned}$$

   we see that the distribution $\tilde{\mathcal{D}}$on Q° spanned by the 7 linearly independent vector fields

   $$\mathscr{L}_{\mathrm{R}}(E_1), \mathscr{L}_{\mathrm{R}}(E_2), \mathscr{L}_{\mathrm{R}}(E_3), \nu((\cdot) \star E_2), L_1, \tilde{L}_2, L_3,$$

with $L_1, \tilde{L}_2, L_3$ as above, is involutive. Moreover $\tilde{\mathcal{D}}$ contains $\mathcal{D}_{\mathrm{R}}|_{Q^\circ}$, which implies $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}|_{Q^\circ}}(q_0) \subset \mathcal{O}_{\tilde{\mathcal{D}}}(q_0)$ and hence $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 7$.

To show the equality here, notice that since $M_0 \cap M^\circ \neq \emptyset$, one has that $O := \pi_{Q,M}^{-1}(M_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an open non-empty subset of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Moreover, because $K_2(x) \neq \beta^2$ on $M_0 \cap M^\circ$, we get that $\nu(A \star E_2)|_q \in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ for all $q \in O$, from which one deduces by Proposition 5.5, case (i) that $\tilde{\mathcal{D}}|_q \subset T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, which then implies $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 7$. This proves one half of case (i) in the statement of this proposition.

b) If (M°, g) has constant curvature, one proves as in case a), by simply changing the roles of M and $\hat{M}$, that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 7$. This finishes the proof of case (i) of this proposition.

For the last case, we assume that neither (M°, g) nor $(\hat{M}^\circ, \hat{g})$ have constant curvature i.e. we have $M^\circ \cap M_0 \neq \emptyset$ and $\hat{M}^\circ \cap \hat{M}_0 \neq \emptyset$.

c) Since $M^\circ \cap M_0 \neq \emptyset$, there is a $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $x_1 \in M_0$. If $\hat{x}_1 \in \hat{M}_0$, we have $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and which implies, as we have shown, that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

Suppose then that $\hat{x}_1 \in \overline{\hat{M}_1}$. Then one may choose a sequence $q'_n = (x'_n, \hat{x}'_n; A'_n) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $q'_n \to q_1$ and $\hat{x}'_n \in \hat{M}_1$. Because M_0 is open, for n large enough one has $(x'_n, \hat{x}'_n) \in (M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Hence $(M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and $\emptyset \neq \hat{M}^\circ \cap \hat{M}_0 \subset \hat{M}_0$, which has been shown to imply that $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and again $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

The proof is complete. □

Remark 5.32 One could adapt the proofs of Propositions 5.29, 5.30 and 5.31 to deal also with the case $\beta = 0$. For example, Proposition 5.29 as formulated already is valid in this case, but the conclusion when $\beta = 0$ could be strengthened to $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 6$. However, since a Riemannian manifold of class $\mathcal{M}_0$ is also locally a Riemannian product, and hence locally a warped product, we prefer to view this special case $\beta = 0$ as part of the subject of Subsection 5.3.2.

5.3.2 Case where both manifolds are Warped Products

Suppose $(M, g) = (I \times N, h_f)$ and $(\hat{M}, \hat{g}) = (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$, where $I, \hat{I} \subset \mathbb{R}$ are open intervals, (N, h) and $(\hat{N}, \hat{h})$ are connected, oriented 2-dimensional Riemannian manifolds and the warping functions $f, \hat{f}$ are smooth and positive everywhere. We write $\frac{\partial}{\partial r}$ for the canonical, positively directed unit vector field on $(\mathbb{R}, s_1)$ and consider it as a vector field on (M, g) and $(\hat{M}, \hat{g})$ as is usual in direct products. Notice that then $\frac{\partial}{\partial r}$ is a g-unit (resp. $\hat{g}$-unit) vector field on M (resp. $\hat{M}$) which is orthogonal to $T|_y N$ (resp. $T|_{\hat{y}}\hat{N}$) for every $(r, y) \in M$ (resp. $(\hat{r}, \hat{y}) \in \hat{N}$). We will prove that starting from any point point $q_0 \in Q = Q(M, \hat{M})$ and if the warping functions $f, \hat{f}$ satisfy extra conditions relative to each other, then the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is either 6- or 8-dimensional. The first case is formulated in the following proposition.

Proposition 5.33 Let $(M,g)=(I\times N,h_f)$, $(\hat{M},\hat{g})=(\hat{I}\times\hat{N},\hat{h}_{\hat{f}})$ be warped products of dimension 3, with $I,\hat{I}\subset\mathbb{R}$ open intervals. Also, let $q_0=(x_0,\hat{x}_0;A_0)\in Q$ be such that if one writes $x_0=(r_0,y_0)$, $\hat{x}_0=(\hat{r}_0,\hat{y}_0)$, then

$$A_0\frac{\partial}{\partial r}\big|_{(r_0,y_0)}=\frac{\partial}{\partial r}\big|_{(\hat{r}_0,y_0)}. \tag{50}$$

holds and

$$\frac{f'(t+r_0)}{f(t+r_0)}=\frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)},\quad \forall t\in(I-r_0)\cap(\hat{I}-\hat{r}_0). \tag{51}$$

Then if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, one has $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=6$.

Proof. For convenience we write $\kappa(r):=\frac{f'(r+r_0)}{f(r+r_0)}=\frac{\hat{f}'(r+\hat{r}_0)}{\hat{f}(r+\hat{r}_0)}$, $r\in(I-r_0)\cap(\hat{I}-\hat{r}_0)=:J$. Let γ be a smooth curve in M defined on some interval containing 0 and such that $\gamma(0)=x_0$ and let $(\gamma(t),\hat{\gamma}(t);A(t))=q_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0)(t)$ be the rolling curve generated by γ starting at q_0 and defined on some (possible smaller) maximal interval containing 0. Write $\gamma(t)=(r(t),\gamma_1(t))$ and $\hat{\gamma}(t)=(\hat{r}(t),\hat{\gamma}_1(t))$ corresponding to the direct products $M=I\times N$ and $\hat{M}=\hat{I}\times\hat{N}$. Define also,

$$\begin{aligned}\zeta(t):=r(t)-r_0,\quad S(t):=&\frac{\partial}{\partial r}\big|_{\gamma(t)},\\ \hat{\zeta}(t):=\hat{r}(t)-\hat{r}_0,\quad \hat{S}(t):=&A(t)^{-1}\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)},\end{aligned}$$

which are vector fields on M along γ. Notice that

$$\begin{aligned}\dot{\zeta}(t)=\dot{r}(t)=&g\big(\dot{\gamma}(t),\frac{\partial}{\partial r}\big|_{\gamma(t)}\big)=g(\dot{\gamma}(t),S(t)),\\ \dot{\hat{\zeta}}(t)=\dot{\hat{r}}(t)=&\hat{g}\big(\dot{\hat{\gamma}}(t),\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big)=\hat{g}\big(A(t)\dot{\gamma}(t),\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big)=g(\dot{\gamma}(t),\hat{S}(t)).\end{aligned}$$

By Proposition 35, Chapter 7, p. 206 in [26], we have

$$\begin{aligned}\nabla_{\dot{\gamma}(t)}\frac{\partial}{\partial r}=&\frac{f'(r(t))}{f(r(t))}\big(\dot{\gamma}(t)-\dot{r}(t)\frac{\partial}{\partial r}\big|_{\gamma(t)}\big),\\ =&\kappa(\zeta(t))\big(\dot{\gamma}(t)-\dot{\zeta}(t)\frac{\partial}{\partial r}\big|_{\gamma(t)}\big),\\ \hat{\nabla}_{\dot{\hat{\gamma}}(t)}\frac{\partial}{\partial r}=&\frac{\hat{f}'\big(\hat{r}(t)\big)}{\hat{f}(\hat{r}(t))}\big(\dot{\hat{\gamma}}(t)-\dot{\hat{r}}(t)\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big),\\ =&\kappa(\hat{\zeta}(t))\big(\dot{\hat{\gamma}}(t)-\dot{\hat{\zeta}}(t)\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big),\end{aligned}$$

i.e.,

$$\begin{aligned}\nabla_{\dot{\gamma}(t)}S(t)=&\kappa(\zeta(t))(\dot{\gamma}(t)-\dot{\zeta}(t)S(t)),\\ \nabla_{\dot{\gamma}(t)}\hat{S}(t)=&A(t)^{-1}\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\frac{\partial}{\partial r}=\kappa(\hat{\zeta}(t))\big(A(t)^{-1}\dot{\hat{\gamma}}(t)-\dot{\hat{\zeta}}(t)A(t)^{-1}\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big),\\ =&\kappa(\hat{\zeta}(t))(\dot{\gamma}(t)-\dot{\hat{\zeta}}(t)\hat{S}(t)).\end{aligned}$$

Let $\rho \in C^\infty(\mathbb{R})$ and $t \mapsto X(t)$ be a vector field along γ and consider a first order ODE

$$\begin{cases} \dot\rho(t) = g(\dot\gamma(t), X(t)), \\ \nabla_{\dot\gamma(t)} X = \kappa(\rho(t))(\dot\gamma(t) - \dot\rho(t)X(t)). \end{cases}$$

By the above we see that the pairs $(\rho, X) = (\zeta, S)$ and $(\rho, X) = (\hat{\zeta}, \hat{S})$ both solve this ODE. Moreover, by assumption $\zeta(0) = 0 = \hat{\zeta}(0)$ and $\hat{S}(0) = A(0)^{-1}\frac{\partial}{\partial r}\big|_{\hat{x}_0} = \frac{\partial}{\partial r}\big|_{x_0} = S(0)$ so these pairs have the same initial conditions and hence $(\zeta, S) = (\hat{\zeta}, \hat{S})$ on the interval where they are both defined. Then,

$$\begin{aligned} r(t) - r_0 =& \hat{r}(t) - \hat{r}_0, \\ A(t)\frac{\partial}{\partial r}\big|_{\gamma(t)} =& \frac{\partial}{\partial r}\big|_{\hat\gamma(t)}, \end{aligned}$$

for all t in the interval where the rolling curve $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)$ is defined. Define

$$Q^*_+ = \big\{q = (x, \hat{x}; A) = ((r, y), (\hat{r}, \hat{y}); A) \in Q \mid r - r_0 = \hat{r} - \hat{r}_0,\ A\frac{\partial}{\partial r}\big|_x = \frac{\partial}{\partial r}\big|_{\hat{x}}\big\}.$$

By the above considerations, as long as the curve is defined,

$$q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) \in Q^*_+,$$

which implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset Q^*_+$. We show that Q^*_+ is a 6-dimensional submanifold of Q. Let $q = (x, \hat{x}; A) = ((r, y), (\hat{r}, \hat{y}); A) \in Q$ such that $A\frac{\partial}{\partial r}\big|_x = \frac{\partial}{\partial r}\big|_{\hat{x}}$. Then for all $\alpha \in \mathbb{R}, X' \in T|_y N$ one has

$$\|X'\|_g^2 + \alpha^2 = \Big\|X' + \alpha\frac{\partial}{\partial r}\big|_x\Big\|_g^2 = \Big\|A(X' + \alpha\frac{\partial}{\partial r}\big|_x)\Big\|_{\hat{g}}^2 = \|AX'\|_{\hat{g}}^2 + 2\hat{g}\big(AX', \alpha\frac{\partial}{\partial r}\big|_{\hat{x}}\big) + \alpha^2.$$

This implies that

$$\begin{aligned} &\|X'\|_g^2 = \|AX'\|_{\hat{g}}^2, \\ &\hat{g}\big(AX', \frac{\partial}{\partial r}\big|_{\hat{x}}\big) = 0, \end{aligned}$$

for all $X' \in T|_y N$. Thus $AT|_y N \perp \frac{\partial}{\partial r}\big|_{\hat{x}}$ and also $A\frac{\partial}{\partial r}\big|_x \perp T|_{\hat{y}}\hat{N}$ by assumption. Define

$$Q_1^+ = \big\{q = (x, \hat{x}; A) = ((r, y), (\hat{r}, \hat{y}); A) \in Q \mid A\frac{\partial}{\partial r}\big|_x = \frac{\partial}{\partial r}\big|_{\hat{x}}\big\},$$

and let $q_1 = (x_1, \hat{x}_1; A_1) = ((r_1, y_1), (\hat{r}_1, \hat{y}_1); A_1) \in Q_1^+$. Choose a local oriented h- and $\hat{h}$-orthonormal frames X_1', X_2' in N around y_1 and $\hat{X}_1', \hat{X}_2'$ in $\hat{N}$ around $\hat{y}_1$. Let the corresponding domains be U' and $\hat{U}'$. Writing $E_1 = \frac{\partial}{\partial r}$, $E_2 = \frac{1}{f}X_1'$, $E_3 = \frac{1}{f}X_2'$ on M and $\hat{E}_1 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{1}{\hat{f}}\hat{X}_1'$, $\hat{E}_3 = \frac{1}{\hat{f}}\hat{X}_2'$ on $\hat{M}$, we see that E_1, E_2, E_3 and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are g- and $\hat{g}$-orthonormal oriented frames and we define

$$\begin{aligned} &\Psi : V := \pi_Q^{-1}((\mathbb{R} \times U') \times (\mathbb{R} \times \hat{U}')) \to \mathrm{SO}(3), \\ &\Psi(x, \hat{x}; A) = [(\hat{g}(AE_i, \hat{E}_j))_i^j]. \end{aligned}$$

This is a chart of Q and clearly

$$\Psi(V\cap Q_1^+)=(\mathbb{R}\times U')\times(\mathbb{R}\times\hat{U}')\times\Big\{\begin{pmatrix}1&0\\0&A'\end{pmatrix}\mid A'\in\mathrm{SO}(2)\Big\}.$$

This shows that $Q_1^+\cap V$ is a 7-dimensional submanifold of Q and hence Q_1^+ is a closed 7-dimensional submanifold of Q. Defining $F:Q_1^+\to\mathbb{R}$ by $F((r,y),(\hat{r},\hat{y});A)=(r-r_0)-(\hat{r}-\hat{r}_0)$, we see that $Q_+^*=F^{-1}(0)$. Once we show that F is a submersion, it follows that Q_+^* is a closed codimension 1 submanifold of Q_1^+ (i.e. $\dim Q_+^*=7-1=6$) and thus it is a 6-dimensional submanifold of Q. Indeed, let $q=(x,\hat{x};A)\in Q_1^+$ and let $\gamma(t)$ be an integral curve of $\frac{\partial}{\partial r}$ starting from x and $\hat{\gamma}(t)=\hat{x}$ a constant path. Let $q(t)=(\gamma(t),\hat{\gamma}(t);A(t))$ be the $\mathcal{D}_{\mathrm{NS}}$-lift of $(\gamma,\hat{\gamma})$ starting from q. Then $\dot{\gamma}(t)=\frac{\partial}{\partial r}\big|_{\gamma(t)}$, $\dot{\hat{\gamma}}(t)=0$ and since $\frac{\partial}{\partial r}$ is a unit geodesic field on M, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{g}\big(A(t)\frac{\partial}{\partial r}\big|_{\gamma(t)},\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big)=\hat{g}\big(A(t)\nabla_{\dot{\gamma}(t)}\frac{\partial}{\partial r},\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big)+\hat{g}\big(A(t)\frac{\partial}{\partial r},\hat{\nabla}_0\frac{\partial}{\partial r}\big|_{\hat{\gamma}(t)}\big)=0.$$

This shows that $q(t)\in Q_1^+$ for all t and in particular, $\mathscr{L}_{\mathrm{NS}}(\frac{\partial}{\partial r}\big|_x)|_q=\dot{q}(0)\in T|_qQ_1^+$. Then if one writes $\gamma(t)=(r(t),\gamma_1(t))$, $\hat{\gamma}(t)=\hat{x}=(\hat{r},\hat{y})$=constant, one has $\dot{r}(t)=1$ and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|_0F(q(t))=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big((r(t)-r_0)-(\hat{r}-\hat{r}_0)\big)=1,$$

i.e., $F_*\mathscr{L}_{\mathrm{NS}}(\frac{\partial}{\partial r}\big|_x)|_q=1$, which shows that F is submersive. (Alternatively, one could have uses the charts Ψ as above to prove this fact.) Since we have shown that $\dim Q_+^*=6$ and $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset Q_+^*$, it follows that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\le 6$. To prove the equality here, we will use the assumption that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$. Take local frames $E_i,\hat{E}_i$ as above near x_1 and $\hat{x}_1$, where $q_1=(x_1,\hat{x}_1;A_1)=((r_1,y_1),(\hat{r}_1,\hat{y}_1);A_1)\in\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. The assumption that $\frac{f'(t+r_0)}{f(t+r_0)}=\frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}$ for all $t\in J$ easily imply that $\frac{f''(t+r_0)}{f(t+r_0)}=\frac{\hat{f}''(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}=:\kappa_2(t)$ for all $t\in J$ as well. Respect to the frames $\star E_1,\star E_2,\star E_3$ and $\hat{\star}\hat{E}_1,\hat{\star}\hat{E}_2,\hat{\star}\hat{E}_3$ one has (see Proposition 42, Chapter 7, p. 210 of [26])

$$R|_{(r,y)}=\begin{pmatrix}-\frac{\sigma(y)}{f(r)^2}+\kappa(r-r_0)^2&0&0\\0&\kappa_2(r-r_0)&0\\0&0&\kappa_2(r-r_0)\end{pmatrix},$$

$$\hat{R}|_{(\hat{r},\hat{y})}=\begin{pmatrix}-\frac{\hat{\sigma}(\hat{y})}{\hat{f}(\hat{r})^2}+\kappa(\hat{r}-\hat{r}_0)^2&0&0\\0&\kappa_2(\hat{r}-\hat{r}_0)&0\\0&0&\kappa_2(\hat{r}-\hat{r}_0)\end{pmatrix},$$

where $\sigma(y)$ and $\hat{\sigma}(\hat{y})$ are the unique sectional curvatures of (N,h) and $(\hat{N},\hat{h})$ at points $y,\hat{y}$. Write

$$-K_2(r,y)=-\frac{\sigma(y)}{f(r)^2}+\kappa(r-r_0),\quad -\hat{K}_2(\hat{r},\hat{y})=-\frac{\hat{\sigma}(\hat{y})}{\hat{f}(\hat{r})^2}+\kappa(\hat{r}-\hat{r}_0).$$

Since $A_1\frac{\partial}{\partial r}\big|_{x_1}=\frac{\partial}{\partial r}\big|_{\hat{x}_1}$, we already know that $A_1E_2|_{x_1}$ and $A_1E_3|_{x_1}$ are in the plane $\mathrm{span}\{\hat{E}_2|_{\hat{x}_1},\hat{E}_3|_{\hat{x}_1}\}$. This and the fact that $r_1-r_0=\hat{r}_1-\hat{r}_0$ imply that

$$\widetilde{\mathsf{Rol}}_{q_1}=\begin{pmatrix}-K_2(x_1)+\hat{K}_2(\hat{x}_1)&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

w.r.t. $\star E_1|_{x_1}, \star E_2|_{x_1}, \star E_3|_{x_1}$. Since $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, it follows from Corollary 4.16 and Remark 4.17 that there is a $q_1 \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, where $\widetilde{\mathsf{Rol}}_{q_1} \neq 0$. Hence there is a neighbourhood O of q_1 in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $\widetilde{\mathsf{Rol}}_q \neq 0$. With respect to local frames $E_i, \hat{E}_i$ as above (taking O smaller if necessary), this means that $K_2(x) \neq \hat{K}_2(\hat{x})$ for all $q = (x, \hat{x}; A) \in O$ and since $\nu(\mathsf{Rol}_q(\star E_1))|_q = (-K_2(x) + \hat{K}_2(\hat{x}))\nu(A \star E_1)|_q$, we have

$$\nu(A \star E_1)|_q \in T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0), \quad \forall q \in O.$$

Hence applying Proposition 5.5 case (i) to the frame $F_1 := E_2$, $F_2 := E_1$, $F_3 := E_3$ implies that the 6 linearly independent vectors (notice that we have $\Gamma^1_{(2,3)} = 0$ in that proposition)

$$\mathscr{L}_{\mathrm{R}}(F_1)|_q, \mathscr{L}_{\mathrm{R}}(F_2)|_q, \mathscr{L}_{\mathrm{R}}(F_3)|_q, \nu(A \star F_2)|_q, L_1|_q, L_3|_q,$$

are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ at $q \in O$, where

$$\begin{aligned} L_1 &= \mathscr{L}_{\mathrm{NS}}(F_1)|_q - \Gamma^1_{(1,2)}(x)\nu(A \star F_3)|_q, \\ L_3 &= \mathscr{L}_{\mathrm{NS}}(F_3)|_q + \Gamma^1_{(1,2)}(x)\nu(A \star F_1)|_q, \end{aligned}$$

with $\Gamma^1_{(1,2)}(x) = g(\nabla_{F_1} F_1 F_2) = g(\nabla_{E_2} E_2, E_1) = -\frac{f'(r)}{f(r)}$ if $x = (r, y)$. Hence $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 6$. $\square$

Remark 5.34 The condition $\mathsf{Rol}_{q_1} \neq 0$ in the proof of the previous proposition was equivalent to the condition $K_2(x_1) \neq \hat{K}_2(\hat{x}_1)$ which again means that if $x_1 = (r_1, y_1)$, $\hat{x}_1 = (\hat{x}_1, \hat{y}_1)$,

$$\frac{\sigma(y_1)}{f(r_1)^2} \neq \frac{\hat{\sigma}(\hat{y}_1)}{\hat{f}(r_1)^2},$$

where $\sigma(y)$ (resp. $\hat{\sigma}(\hat{y})$) is the sectional curvature of (N, h) at $y \in N$ (resp. of $(\hat{N}, \hat{h})$ at $\hat{y} \in \hat{N}$).

Remark 5.35 To show that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 6$ under the assumptions of the proposition, we showed that if $q = (x, \hat{x}; A) \in Q^*_+$, then $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q)(t) \in Q^*_+$ for any path γ starting from x. For this we basically used the uniqueness of the solutions of an ODE. Alternatively, one could have proceeded exactly in the same way as in the proof of Proposition 5.29. To this end, one defines as there $h_1, h_1 : Q \to \mathbb{R}$ and also $F : Q \to \mathbb{R}$ as above as

$$h_1(q) = \hat{g}(AE_1, \hat{E}_2), \quad h_2(q) = \hat{g}(AE_3, \hat{E}_2), \quad F(q) = (r - r_0) - (\hat{r} - \hat{r}_0).$$

Write $H = (h_1, h_2, F) : Q \to \mathbb{R}^3$, $Q^* := H^{-1}(0)$ and $Q = Q^*_+ \cup Q^*_-$ where Q^*_+ (resp. Q^*_-) consists of all $q = (x, \hat{x}; A) \in Q^*$ where $A\frac{\partial}{\partial r} = +\frac{\partial}{\partial r}$ (resp. $A\frac{\partial}{\partial r} = -\frac{\partial}{\partial r}$). Then, for all $q \in Q^*_+$,

$$H_* \nu(A \star E_1)|_q = (0, -1, 0), \quad H_* \nu(A \star E_3)|_q = (1, 0, 0), \quad H_* \mathscr{L}_{\mathrm{NS}}(\frac{\partial}{\partial r}, 0)|_q = (0, 0, 1),$$

which shows (again) that Q_+^* is a 6-dimensional closed submanifold of Q (and so is Q^*) while w.r.t. orthonormal bases E_1, E_2, E_3, $\hat{E}_1, \hat{E}_2, \hat{E}_3$, where $E_2 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{\partial}{\partial \hat{r}}$, one has for $q = (x, \hat{x}; A) \in Q_+^*$, since $x = (r, y)$, $\hat{x} = (\hat{r}, \hat{y})$ with $r - r_0 = \hat{r} - \hat{r}_0 =: t$,

$$
\begin{aligned}
\mathscr{L}_{\mathrm{R}}(E_1)|_q h_1 &= \hat{g}(A(\Gamma^1_{(1,2)}E_2 - \Gamma^1_{(3,1)}E_3), \hat{E}_2) + \hat{g}(AE_1, -\hat{\Gamma}^1_{(1,2)}AE_1) \\
&= -\frac{f'(r)}{f(r)} + \frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})} = -\frac{f'(t+r_0)}{f(t+r_0)} + \frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)} = 0, \\
\mathscr{L}_{\mathrm{R}}(E_1)|_q h_2 &= \Gamma^1_{(3,1)}\hat{g}(AE_1, \hat{E}_2) + \hat{g}(AE_3, -\hat{\Gamma}^1_{(1,2)}AE_1) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q h_1 &= -\Gamma^2_{(3,1)}\hat{g}(AE_3, \hat{E}_2) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q h_2 &= \Gamma^2_{(3,1)}\hat{g}(AE_1, \hat{E}_2) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_3)|_q h_1 &= \mathscr{L}_{\mathrm{R}}(E_3)|_q h_2 = 0, \\
\mathscr{L}_{\mathrm{R}}(E_1)|_q F &= \mathscr{L}_{\mathrm{R}}(E_2)|_q F = \mathscr{L}_{\mathrm{R}}(E_3)|_q F = 0,
\end{aligned}
$$

hence $\mathcal{D}_{\mathrm{R}}|_q \subset T|_q Q_+^*$ for all $q \in Q_+^*$. This obviously implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \subset Q_+^*$ for all $q \in Q_+^*$ and thus $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \leq \dim Q_+^* = 6$.

For the following proposition we introduce some more notations,

$$
\begin{aligned}
Q_0 :=& \{q = ((r,y),(\hat{r},\hat{y}); A) \in Q \mid A\frac{\partial}{\partial r}\big|_{(r,y)} \neq \pm \frac{\partial}{\partial r}\big|_{(\hat{r},\hat{y})}\}, \\
Q_1^+ :=& Q \backslash Q_0 = \{q = (x, \hat{x}; A) \in Q \mid A\frac{\partial}{\partial r}\big|_{(r,y)} = +\frac{\partial}{\partial r}\big|_{(\hat{r},\hat{y})}\}, \\
Q_1^- :=& Q \backslash Q_0 = \{q = (x, \hat{x}; A) \in Q \mid A\frac{\partial}{\partial r}\big|_{(r,y)} = -\frac{\partial}{\partial r}\big|_{(\hat{r},\hat{y})}\}, \\
Q_1 :=& Q_1^+ \cup Q_1^-, \\
S_1^+ :=& \{q = ((r,y),(\hat{r},\hat{y}); A) \in Q_1^+ \mid \frac{f'(r)}{f(r)} = +\frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})}\}, \\
S_1^- :=& \{q = ((r,y),(\hat{r},\hat{y}); A) \in Q_1^- \mid \frac{f'(r)}{f(r)} = -\frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})}\}, \\
S_1 :=& S_1^+ \cup S_1^-.
\end{aligned}
$$

We have that Q decomposes into the disjoint union

$$
Q = S_1 \cup (Q \backslash S_1) = S_1 \cup (Q_1 \backslash S_1) \cup Q_0.
$$

Proposition 5.36 Let $(M, g) = (I \times N, h_f)$ and $(\hat{M}, \hat{g}) = (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}})$, be warped products with $I, \hat{I} \subset \mathbb{R}$ open intervals and suppose that there is a constant $K \in \mathbb{R}$ such that

$$
\frac{f''(r)}{f(r)} = -K = \frac{\hat{f}''(\hat{r})}{\hat{f}(\hat{r})}, \quad \forall (r, \hat{r}) \in I \times \hat{I}.
$$

Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and write $M^\circ := \pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $\hat{M}^\circ := \pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Assuming that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$, we have the following cases:

(i) If $q_0 \in S_1$, then $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$;

(ii) If $q_0 \in Q\backslash S_1$ and if only one of (M°, g) or $(\hat{M}^\circ, \hat{g})$ has constant curvature, then $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$;

(iii) Otherwise $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 8$.

Proof. As in the proof of Proposition 5.33 (see also Remark 5.35) it is clear that Q_1 is a closed 7-dimensional closed submanifolds of Q and Q_1^-, Q_1^+ are disjoint open and closed submanifolds of Q_1. Also, S_1, S_1^+, S_1^- are closed subsets of Q_1. Let us begin with the case where $q_0 \in S_1^+$. Writing $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$ and defining $w(t) := \frac{f'(t+r_0)}{f(t+r_0)} - \frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}$, we see that for all $t \in (I - r_0) \cap (\hat{I} - \hat{r}_0)$,

$$w'(t) = \underbrace{\frac{f''(t+r_0)}{f(t+r_0)}}_{=-K} - \Big(\frac{f'(t+r_0)}{f(t+r_0)}\Big)^2 - \underbrace{\frac{\hat{f}''(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}}_{=-K} + \Big(\frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}\Big)^2,$$

i.e.,

$$w'(t) = -w(t)\Big(\frac{f'(t+r_0)}{f(t+r_0)} + \frac{\hat{f}'(t+\hat{r}_0)}{\hat{f}(t+\hat{r}_0)}\Big), \quad w(0) = 0.$$

This shows that $w(t) = 0$ for all $t \in (I - r_0) \cap (\hat{I} - \hat{r}_0)$ and hence the assumptions of Proposition 5.33 have been met. Thus $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$. On the other hand, if $q_0 = (x_0, \hat{x}_0; A_0) \in S_1^-$ and $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$, define $\hat{f}^\vee(t) := \hat{f}(-t)$, $\hat{I}^\vee := -\hat{I}$ and notice that $\varphi : (\hat{I} \times \hat{N}, \hat{h}_{\hat{f}}) \to (\hat{I}^\vee \times \hat{N}, \hat{h}_{\hat{f}^\vee}) =: (\hat{M}^\vee, \hat{g}^\vee)$ given by $(\hat{y}, \hat{r}) \mapsto (\hat{y}, -\hat{r})$ is an isometry, which induces a diffeomorphism $\Phi : Q \to Q(M, \hat{M}^\vee)$ by $(x, \hat{x}; A) \mapsto (x, \varphi(\hat{x}); \varphi_*|_{\hat{x}} \circ A)$ which preserves the respective rolling distributions and orbits: $\Phi_*(\mathcal{D}_{\mathrm{R}}|_q) = \mathcal{D}_{\mathrm{R}}^\vee|_{\Phi(q)}$, $\Phi(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}^\vee}(\Phi(q))$, the notation being clear here. But now $\Phi(A_0) = \varphi_*(A_0 \frac{\partial}{\partial r}) = -\varphi_* \frac{\partial}{\partial r} = \frac{\partial}{\partial r}$ and since $q_0^\vee := \Phi(q_0) = ((r_0, y_0), (-\hat{r}_0, \hat{y}_0); \varphi_* \circ A_0)$,

$$\frac{(f^\vee)'(-\hat{r}_0)}{f^\vee(-\hat{r}_0)} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f^\vee(t - \hat{r}_0)}{\hat{f}(\hat{r}_0)} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(\hat{r}_0 - t)}{\hat{f}(\hat{r}_0)} = -\frac{f'(\hat{r}_0)}{\hat{f}(\hat{r}_0)} = \frac{f'(r_0)}{f(r_0)}.$$

Thus $\Phi(q_0)$ belongs to the set S_1^+ of $Q(M, \hat{M}^\vee)$ (which corresponds by Φ to S_1^- of Q) and thus the above argument implies that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}^\vee}(\Phi(q_0)) = 6$ and therefore $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = 6$. Hence we have proven (i). We next deal with the case where $q_0 \in Q\backslash S_1$. Up until the second half of the proof, where we introduce the sets $M_0, M_1, \hat{M}_0, \hat{M}_1$, we assume that the choice of $q_0 \in Q\backslash S_1$ is not fixed (and hence M°, $\hat{M}^\circ$ are not defined yet). So let $q_0 = (x_0, \hat{x}_0; A_0) = ((r_0, y_0), (\hat{r}_0, \hat{y}_0); A_0) \in Q\backslash S_1$ and choose some orthonormal frame X_1, X_3 (resp. $\hat{X}_1, \hat{X}_3$) on N (resp. $\hat{N}$) defined on an open neighbourhood U' of y_0 (resp. $\hat{U}'$ of $\hat{y}_0$) and consider them, in the natural way, as vector fields on M (resp. $\hat{M}$). Moreover, assume that $X_1, \frac{\partial}{\partial r}, X_3$ (resp. $\hat{X}_1, \frac{\partial}{\partial r}, \hat{X}_3$) is oriented. Writing $E_1 = \frac{1}{f} X_1$, $E_2 = \frac{\partial}{\partial r}$, $E_3 = \frac{1}{f} X_3$, and $\hat{E}_1 = \frac{1}{\hat{f}} \hat{X}_1$, $\hat{E}_2 = \frac{\partial}{\partial r}$, $\hat{E}_3 = \frac{1}{\hat{f}} \hat{X}_3$, we get positively oriented orthonormal frames of M and $\hat{M}$, defined on $U := I \times U'$, $\hat{U} := \hat{I} \times \hat{U}'$, respectively. Then we have, by [26], Chapter 7, Proposition 42 (one should pay attention that there the definition of the curvature tensor differs by sign to the definition used here) that with respect to the frames $\star E_1, \star E_2, \star E_3$ and $\hat{\star}\hat{E}_1, \hat{\star}\hat{E}_2, \hat{\star}E_3$,

$$R = \begin{pmatrix} -K & 0 & 0 \\ 0 & \frac{-\sigma+(f')^2}{f^2} & 0 \\ 0 & 0 & -K \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} -K & 0 & 0 \\ 0 & \frac{-\hat{\sigma}+(\hat{f}')^2}{\hat{f}^2} & 0 \\ 0 & 0 & -K \end{pmatrix},$$

where $\sigma(y)$ and $\hat{\sigma}(\hat{y})$ are the unique sectional (or Gaussian) curvatures of (N, h) and $(\hat{N}, \hat{h})$ at points $y, \hat{y}$. Write $-K_2 := \frac{-\sigma+(f')^2}{f^2}$ and $-\hat{K}_2 := \frac{-\hat{\sigma}+(\hat{f}')^2}{\hat{f}^2}$. We now take an open neighbourhood $\tilde{O}$ of q_0 in Q according to the following cases:

(a) If $q_0 \in Q_0$, we assume that $\tilde{O} \subset Q_0 \cap \pi_Q^{-1}(U \times \hat{U})$.

(b) If $q_0 \in Q_1^+ \backslash S_1$ (resp. $q_0 \in Q_1^- \backslash S_1$) we assume that $\tilde{O} \subset \pi_Q^{-1}(U \times \hat{U}) \backslash (S_1 \cup Q_1^-)$ (resp. $\tilde{O} \subset \pi_Q^{-1}(U \times \hat{U}) \backslash (S_1 \cup Q_1^+)$).

Write $\tilde{O}_0 := \tilde{O} \cap Q_0$. Thus in case (a) one has $\tilde{O} = \tilde{O}_0 \ni q_0$ while in case (b) one has $\tilde{O} = \tilde{O}_0 \cup (\tilde{O} \cap (Q_1^\pm \backslash S_1))$, as a disjoint union, and $q_0 \notin \tilde{O}_0$, the "$\pm$" depending on the respective situation. Moreover, if the case (b) occurs, we assume that $q_0 \in Q_1^+ \backslash S_1$ since the case where $q_0 \in Q_1^- \backslash S_1$ is handled in a similar way. We will still shrink $\tilde{O}$ around q_0 whenever convenient and always keep in mind that $\tilde{O}_0 = \tilde{O} \cap Q_0$ even after the shrinking. Notice that this shrinking does not change the properties in (a) and (b) above. Moreover, [26], Chapter 7, Proposition 35 implies that if Γ, $\hat{\Gamma}$ are connection tables w.r.t. E_1, E_2, E_3 and $\hat{E}_1, \hat{E}_2, \hat{E}_3$, respectively,

$$\Gamma = \begin{pmatrix} 0 & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & 0 \end{pmatrix}, \quad \hat{\Gamma} = \begin{pmatrix} 0 & 0 & -\hat{\Gamma}^1_{(1,2)} \\ \hat{\Gamma}^1_{(3,1)} & \hat{\Gamma}^2_{(3,1)} & \hat{\Gamma}^3_{(3,1)} \\ \hat{\Gamma}^1_{(1,2)} & 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} W(\Gamma^1_{(1,2)}) =&0, \quad \forall W \in E_2^\perp, \\ \hat{W}(\hat{\Gamma}^1_{(1,2)}) =&0, \quad \forall \hat{W} \in \hat{E}_2^\perp, \end{aligned}$$

since $\Gamma^1_{(1,2)}(r, y) = -\frac{f'(r)}{f(r)}$ and $\hat{\Gamma}^1_{(1,2)}(\hat{r}, \hat{y}) = -\frac{\hat{f}'(\hat{r})}{\hat{f}(\hat{r})}$. Actually one even has $\Gamma^2_{(3,1)} = 0$ and $\hat{\Gamma}^2_{(3,1)} = 0$, but we do not use this fact; one could for example rotate E_1, E_3 (resp. $\hat{E}_1, \hat{E}_3$) between them, in a non-constant way, to destroy this property. The fact that $AE_2|_x \neq \pm\hat{E}_2|_{\hat{x}}$ for $q = (x, \hat{x}; A) \in Q_0$ is equivalent to the fact that the intersection $(AE_2^\perp|_x) \cap \hat{E}_2^\perp|_{\hat{x}}$ is non-trivial for all $q = (x, \hat{x}; A) \in Q_0$. Therefore, by shrinking $\tilde{O}$ around q_0 if necessary, we may find a smooth functions $\theta, \hat{\theta} : \tilde{O}_0 \to \mathbb{R}$ such that this intersection is spanned by $AZ_A = \hat{Z}_A$, where

$$\begin{aligned} Z_A :=& -\sin(\theta(q))E_1|_x + \cos(\theta(q))E_3|_x, \\ \hat{Z}_A :=& -\sin(\hat{\theta}(q))\hat{E}_1|_{\hat{x}} + \cos(\hat{\theta}(q))\hat{E}_3|_{\hat{x}}. \end{aligned}$$

We also define

$$\begin{aligned} X_A :=& \cos(\theta(q))E_1|_x + \sin(\theta(q))E_3|_x, \\ \hat{X}_A :=& \cos(\hat{\theta}(q))\hat{E}_1|_{\hat{x}} + \sin(\hat{\theta}(q))\hat{E}_3|_{\hat{x}}. \end{aligned}$$

To unburden the formulas, we write from now on usually $s_\tau := \sin(\tau(q))$, $c_\tau := \cos(\tau(q))$ if $\tau : \tilde{V} \to \mathbb{R}$ is some function, $\tilde{V} \subset Q$, and the point $q \in \tilde{V}$ is clear from the context. Since $X_A, E_2|_x, Z_A$ (resp. $\hat{X}_A, \hat{E}_2|_{\hat{x}}, \hat{Z}_A$) form an orthonormal frame for every $q = (x, \hat{x}; A) \in \tilde{O}_0$ and because $A(Z_A^\perp) = \hat{Z}_A^\perp$, it follows that there is a smooth $\phi : \tilde{O}_0 \to \mathbb{R}$ such that

$$
\begin{aligned}
AX_A =& c_\phi \hat{X}_A + s_\phi \hat{E}_2 = c_\phi(c_{\hat{\theta}} \hat{E}_1 + s_{\hat{\theta}} \hat{E}_3) + s_\phi \hat{E}_2, \\
AE_2 =& - s_\phi \hat{X}_A + c_\phi \hat{E}_2 = -s_\phi(c_{\hat{\theta}} \hat{E}_1 + s_{\hat{\theta}} \hat{E}_3) + c_\phi \hat{E}_2, \\
AZ_A =& \hat{Z}_A.
\end{aligned}
$$

In particular, for all $q = (x, \hat{x}; A) \in \tilde{O}_0$, one has $\hat{g}(AZ_A, \hat{E}_2) = 0$. Formulas in Eq. (45) on page 58 hold with $\Gamma^1_{(2,3)} = 0$ and $Y = \hat{E}_2$. Since they are very useful in computations, we will now derive three relations, two of which simplify Eq. (45), and all of which play an important role later on in the proof. Differentiating the identity $\hat{g}(AZ_A, \hat{E}_2) = 0$ with respect to $\mathscr{L}_{\mathrm{R}}(X_A)|_q$, $\mathscr{L}_{\mathrm{R}}(E_2)|_q$ and $\mathscr{L}_{\mathrm{R}}(Z_A)|_q$, one at a time, yields on $\tilde{O}_0$,

$$
\begin{aligned}
0 =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(X_A)Z_{(\cdot)}, \hat{E}_2) + \hat{g}(AZ_A, \hat{\nabla}_{AX_A}\hat{E}_2) \\
=& (-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})\hat{g}(AX_A, \hat{E}_2) + \hat{g}(\hat{Z}_A, -c_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A) \\
=& s_\phi(-\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}), \\
0 =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(E_2)Z_{(\cdot)}, \hat{E}_2) + \hat{g}(AZ_A, \hat{\nabla}_{AE_2}\hat{E}_2) \\
=& (-\mathscr{L}_{\mathrm{R}}(Y)|_q\theta + \Gamma^2_{(3,1)})\hat{g}(AX_A, \hat{E}_2) + \hat{g}(\hat{Z}_A, s_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A) \\
=& s_\theta(-\mathscr{L}_{\mathrm{R}}(Y)|_q\theta + \Gamma^2_{(3,1)}), \\
0 =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(Z_A)Z_{(\cdot)}, \hat{E}_2) + \hat{g}(AZ_A, \hat{\nabla}_{AZ_A}\hat{E}_2) \\
=& (-\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})\hat{g}(AX_A, \hat{E}_2) \\
&+ \Gamma^1_{(1,2)}\hat{g}(AE_2, \hat{E}_2) + \hat{g}(\hat{Z}_A, -\hat{\Gamma}^1_{(1,2)}\hat{Z}_A) \\
=& s_\phi(-\mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)}) + c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}.
\end{aligned}
$$

Define

$$
\lambda(q) := \mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta + s_\theta\Gamma^1_{(3,1)} - c_\theta\Gamma^3_{(3,1)}, \quad q \in \tilde{O}_0,
$$

which is a smooth function on $\tilde{O}_0$. Since $\sin(\phi(q)) = 0$ would imply that $AE_2 = \pm\hat{E}_2$, we have $\sin(\phi(q)) \neq 0$ on $\tilde{O}_0 \subset Q_0$ and hence we get

$$
\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta &= c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q\theta &= \Gamma^2_{(3,1)}, \\
s_\phi\lambda &= c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}.
\end{aligned}
$$

These formulas, along with $\Gamma^1_{(2,3)} = 0$, simplify Eq. (45) to

$$
\begin{aligned}
&\mathscr{L}_{\mathrm{R}}(X_A)|_qX_{(\cdot)} = \Gamma^1_{(1,2)}E_2, \quad \mathscr{L}_{\mathrm{R}}(E_2)|_qX_{(\cdot)} = 0, \quad \mathscr{L}_{\mathrm{R}}(Z_A)|_qX_{(\cdot)} = \lambda Z_A, \\
&\mathscr{L}_{\mathrm{R}}(X_A)|_qE_2 = -\Gamma^1_{(1,2)}X_A, \quad \mathscr{L}_{\mathrm{R}}(E_2)|_qE_2 = 0, \quad \mathscr{L}_{\mathrm{R}}(Z_A)|_qE_2 = -\Gamma^1_{(1,2)}Z_A, \\
&\mathscr{L}_{\mathrm{R}}(X_A)|_qZ_{(\cdot)} = 0, \quad \mathscr{L}_{\mathrm{R}}(E_2)|_qZ_{(\cdot)} = 0, \quad \mathscr{L}_{\mathrm{R}}(Z_A)|_qZ_{(\cdot)} = -\lambda X_A + \Gamma^1_{(1,2)}E_2, \quad (52)
\end{aligned}
$$

at $q \in \tilde{O}_0$. We use these in the rest of the proof without further mention. Notice that, for any $q = (x, \hat{x}; A) \in (Q_1^+ \backslash S_1) \cap \tilde{O}$, and any sequence (which exist as $Q_1 \cap \tilde{O}$ is a nowhere dense subset of $\tilde{O}$) $q_n \in \tilde{O}_0$, $q_n \to q$, we have $\cos(\phi(q_n)) \to \cos(\phi(q)) = 1$, hence $0 \neq \sin(\phi(q_n)) \to 0$. Because

$$\lim_{n\to\infty}(c_\phi \Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)})(q_n) = (c_\phi \Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)})(q) = \Gamma^1_{(1,2)}(x) - \hat{\Gamma}^1_{(1,2)}(\hat{x}) \neq 0,$$

as $q \in Q_1^+ \backslash S_1$, we get

$$\lim_{n\to\infty} \big(\sin(\phi(q_n))\lambda(q_n)\big) \neq 0, \quad \lim_{n\to\infty} \sin(\phi(q_n)) = 0,$$

which implies that $\lim_{n\to\infty} \lambda(q_n) = \pm\infty$. In particular, we see that, even after shrinking $\tilde{O}$, one cannot extend the definition of θ in a smooth, or even C^1, way onto $\tilde{O}$, since if this were possible, the definition of λ above would imply that λ is continuous on $\tilde{O}$ and hence the above sequences $\lambda(q_n)$ would be bounded. This fact about the unboudedness of $\lambda(q)$ as q approaches $(Q_1^+ \backslash S_1) \cap \tilde{O}$ will be used later. To get around this problem, we will be working for a while uniquely on $\tilde{O}_0$.

Define on $\tilde{O}_0$ a 5-dimensional smooth distribution Δ spanned by

$$\mathscr{L}_{\mathrm{R}}(E_1)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(E_3)|_q, \nu(A \star E_2)|_q, \nu(A \star X_A)|_q, \quad q \in \tilde{O}_0.$$

We will proceed to show that the Lie algebra $\mathrm{Lie}(\Delta)$ spans at every point of $q \in \tilde{O}_0$ a 8-dimensional distribution $\mathrm{Lie}(\Delta)|_q$ which is then necessarily involutive. Notice that we consider VF^k_Δ, $k = 1, 2, \dots$ and $\mathrm{Lie}(\Delta)$ as $C^\infty(\tilde{O}_0)$-modules. Since $\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \mathscr{L}_{\mathrm{R}}(E_2), \mathscr{L}_{\mathrm{R}}(Z_{(\cdot)})$ span $\mathcal{D}_{\mathrm{R}}$ on $\tilde{O}_0$, they generate the module $\mathrm{VF}_{\mathcal{D}_{\mathrm{R}}|_{\tilde{O}_0}}$ and hence $\mathrm{Lie}(\mathcal{D}_{\mathrm{R}}|_{\tilde{O}_0})$. Moreover, the brackets

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \mathscr{L}_{\mathrm{R}}(E_2)]|_q =& -\Gamma^1_{(1,2)}\mathscr{L}_{\mathrm{R}}(X_A)|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_2), \mathscr{L}_{\mathrm{R}}(Z_{(\cdot)})]|_q =& \Gamma^1_{(1,2)}\mathscr{L}_{\mathrm{R}}(Z_A)|_q - K_1^{\mathsf{Rol}}\nu(A \star X_A)|_q - \alpha\nu(A \star E_2)|_q, \\
[\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}), \mathscr{L}_{\mathrm{R}}(X_{(\cdot)})]|_q =& \lambda\mathscr{L}_{\mathrm{R}}(Z_A)|_q - \alpha\nu(A \star X_A)|_q - K_2^{\mathsf{Rol}}\nu(A \star E_2)|_q,
\end{aligned}$$

along with the definition of X_A, Z_A, show that $\mathrm{VF}^2_{\mathcal{D}_{\mathrm{R}}|_{\tilde{O}_0}} \subset \mathrm{VF}_\Delta$.

The first three Lie brackets in Proposition 5.5 case (ii) show that VF^2_Δ contains vector fields L_1, L_3 given by $L_1|_q = \mathscr{L}_{\mathrm{NS}}(E_1)|_q - \Gamma^1_{(1,2)}\nu(A\star E_3)|_q$, $L_3|_q = \mathscr{L}_{\mathrm{NS}}(E_3)|_q + \Gamma^1_{(1,2)}\nu(A\star E_1)|_q$, and also $L_2|_q$, which in this setting is just the zero-vector field on $\tilde{O}_0$. We define $F_X|_q := c_\theta L_1|_q + s_\theta L_3|_q$ and $F_Z|_q := -s_\theta L_1|_q + c_\theta L_3|_q - \Gamma^1_{(1,2)}\nu(A \star X_A)|_q$, hence $F_X, F_X \in \mathrm{VF}^2_\Delta$ and one easily sees that they simplify to

$$\begin{aligned}
F_X|_q =& \mathscr{L}_{\mathrm{NS}}(X_A)|_q - \Gamma^1_{(1,2)}\nu(A \star Z_A)|_q, \\
F_Z|_q =& \mathscr{L}_{\mathrm{NS}}(Z_A)|_q.
\end{aligned}$$

It is clear that the vector fields

$$\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}), \mathscr{L}_{\mathrm{R}}(E_2), \mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}), \nu((\cdot) \star E_2), \nu((\cdot) \star X_{(\cdot)}), F_X, F_Z,$$

span the same $C^\infty(\tilde{O}_0)$-submodule of VF^2_Δ as do

$$\mathscr{L}_{\mathrm{R}}(E_1), \mathscr{L}_{\mathrm{R}}(E_2), \mathscr{L}_{\mathrm{R}}(E_3), \nu((\cdot) \star E_2), \nu((\cdot) \star X_{(\cdot)}), L_1, L_3.$$

We now want to find generators of VF^2_Δ. By what we have already done and said, it remains us to compute need to prove that the Lie-brackets between the 4 vector fields

$$\mathscr{L}_{\mathrm{R}}(X_A)|_q, \mathscr{L}_{\mathrm{R}}(E_2)|_q, \mathscr{L}_{\mathrm{R}}(Z_A)|_q, \nu(A \star E_2)|_q,$$

and $\nu((\cdot)\star X_{(\cdot)})|_q$. Since we will have to derivate X_A, it follows that the derivatives of θ will also appear. That is why we first compute with respect to all the (pointwise linearly independent) vectors that appear above. As a first step, compute

$$\begin{aligned}
F_X|_q Z_{(\cdot)} =&(-F_X|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})X_A,\\
F_Z|_q Z_{(\cdot)} =&\mathscr{L}_{\mathrm{NS}}(Z_A)|_q Z_{(\cdot)} = (-F_Z|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})X_A + \Gamma^1_{(1,2)}E_2.
\end{aligned}$$

Knowing already $\mathscr{L}_{\mathrm{R}}(X_A)|_q\theta, \mathscr{L}_{\mathrm{R}}(Y)|_q\theta, \mathscr{L}_{\mathrm{R}}(Z_A)|_q\theta$, we derivate the identity

$$\hat{g}(AZ_A, \hat{E}_2) = 0,$$

w.r.t. $\nu(A\star E_2)|_q, \nu(A\star X_A)|_q, F_X|_q, F_Z|_q$, which gives (notice that the derivative of $\hat{E}_2$ with respect to these vanishes)

$$\begin{aligned}
0 =&\hat{g}(A(\star E_2)Z_A - \nu(A\star E_2)|_q\theta A X_A, \hat{E}_2)\\
=&(1-\nu(A\star E_2))\hat{g}(AX_A, \hat{E}_2) = s_\phi(1-\nu(A\star E_2)),\\
0 =&\hat{g}(A(\star X_A)Z_A - \nu(A\star X_A)|_q\theta A X_A, \hat{E}_2)\\
=&-\hat{g}(AE_2, \hat{E}_2) - \nu(A\star X_A)|_q\theta\hat{g}(AX_A, \hat{E}_2)\\
=&-c_\phi - s_\phi\nu(A\star X_A)|_q\theta,\\
0 =&\hat{g}(-\Gamma^1_{(1,2)}A(\star Z_A)Z_A, \hat{E}_2) + (-F_X|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)})\hat{g}(AX_A, \hat{E}_2)\\
=&s_\phi(-F_X|_q\theta + c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)}),\\
0 =&(-F_Z|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)})\hat{g}(AX_A, E_2) + \Gamma^1_{(1,2)}\hat{g}(AE_2, \hat{E}_2)\\
=&s_\phi(-F_Z|_q\theta - s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)}) + c_\phi\Gamma^1_{(1,2)},
\end{aligned}$$

and since $s_\phi \neq 0$ on $\tilde{O}_0$,

$$\begin{aligned}
\nu(A\star E_2)|_q\theta =&1,\\
\nu(A\star X_A)|_q\theta =&-\cot(\phi),\\
F_X|_q\theta =&c_\theta\Gamma^1_{(3,1)} + s_\theta\Gamma^3_{(3,1)},\\
F_Z|_q\theta =&-s_\theta\Gamma^1_{(3,1)} + c_\theta\Gamma^3_{(3,1)} + \cot(\phi)\Gamma^1_{(1,2)}.
\end{aligned}$$

These simplify the above formulas to

$$\begin{aligned}
F_X|_q Z_{(\cdot)} =&0,\\
F_Z|_q Z_{(\cdot)} =&\mathscr{L}_{\mathrm{NS}}(Z_A)|_q Z_{(\cdot)} = -\cot(\phi)X_A + \Gamma^1_{(1,2)}E_2,
\end{aligned}$$

and moreover it is now easy to see that for $q\in\tilde{O}_0$,

$$\begin{aligned}
F_X|_q X_{(\cdot)} =&\Gamma^1_{(1,2)}E_2, \quad F_X|_q E_2 = -\Gamma^1_{(1,2)}X_A,\\
F_Z|_q X_{(\cdot)} =&\cot(\phi)\Gamma^1_{(1,2)}Z_A, \quad F_Z|_q E_2 = -\Gamma^1_{(1,2)}Z_A.
\end{aligned}$$

The Lie brackets

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}),\nu((\cdot)\star X_{(\cdot)})]|_q =&\cot(\phi)\mathscr{L}_{\mathrm{R}}(Z_A)|_q-\mathscr{L}_{\mathrm{NS}}(A\star(\star X_A)X_A)|_q+\Gamma^1_{(1,2)}\nu(A\star E_2)|_q\\
=&\cos(\phi)\mathscr{L}_{\mathrm{R}}(Z_A)|_q+\Gamma^1_{(1,2)}\nu(A\star E_2)|_q,\\
[\mathscr{L}_{\mathrm{R}}(E_2),\nu((\cdot)\star X_{(\cdot)})]|_q =&-\mathscr{L}_{\mathrm{NS}}(A(\star X_A)E_2)|_q+\nu(A\star 0)|_q=F_Z|_q-\mathscr{L}_{\mathrm{R}}(Z_A)|_q,\\
[\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}),\nu((\cdot)\star X_{(\cdot)})]|_q =&-\cot(\phi)\mathscr{L}_{\mathrm{R}}(X_A)|_q-\mathscr{L}_{\mathrm{NS}}(A\star(\star X_A)Z_A)|_q+\nu(A\star(\lambda Z_A)),\\
=&-\cot(\phi)\mathscr{L}_{\mathrm{R}}(X_A)|_q+\mathscr{L}_{\mathrm{R}}(E_2)|_q-(\mathscr{L}_{\mathrm{NS}}(E_2)|_q-\lambda\nu(A\star Z_A)|_q),\\
[\nu(A\star E_2),\nu((\cdot)\star X_{(\cdot)})]|_q =&\nu(A[\star E_2,\star X_A]_{\mathfrak{so}})|_q+\nu(A\star Z_A)|_q=0,
\end{aligned}$$

show that if one defines

$$F_Y|_q:=\mathscr{L}_{\mathrm{NS}}(E_2)|_q-\lambda\nu(A\star Z_A)|_q,$$

then one may write

$$[\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}),\nu((\cdot)\star X_{(\cdot)})]|_q=-\cot(\phi)\mathscr{L}_{\mathrm{R}}(X_A)|_q+\mathscr{L}_{\mathrm{R}}(E_2)|_q-F_Y|_q,$$

and hence we have shown that VF^2_Δ is generated by vector fields

$$\mathscr{L}_{\mathrm{R}}(X_{(\cdot)}),\mathscr{L}_{\mathrm{R}}(E_2),\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)}),\nu((\cdot)\star E_2),\nu((\cdot)\star X_{(\cdot)}),F_X,F_Y,F_Z,$$

which are all pointwise linearly independent on $\tilde{O}_0$.

Next we will proceed to show that the VF^2_Δ generated by the above 8 vector fields is in fact involutive, which then establishes that $\mathrm{Lie}(\Delta)=\mathrm{VF}^2_\Delta$. At first, the last 9 Lie brackets in Proposition 5.5 (recall that we have $\Gamma^1_{(2,3)}=0$) show that $[F_Z,F_X]$ and the brackets of $\mathscr{L}_{\mathrm{R}}(X_{(\cdot)})$, $\mathscr{L}_{\mathrm{R}}(E_2)$, $\mathscr{L}_{\mathrm{R}}(Z_{(\cdot)})$, $\nu((\cdot)\star E_2)$, with F_X and F_Z all belong to VF^2_Δ as well as do

$$\begin{aligned}
[F_X,\nu((\cdot)\star X_{(\cdot)})]|_q =&-\mathscr{L}_{\mathrm{NS}}(-\cot(\phi)Z_A)|_q+\nu(A\star(\mathscr{L}_{\mathrm{NS}}(X_A)|_qX_{(\cdot)}))|_q\\
&-\Gamma^1_{(1,2)}\nu(A[\star Z_A,\star X_A]_{\mathfrak{so}}+\nu(A\star Z_A)|_qX_{(\cdot)}-\cot(\phi)A\star X_A)|_q\\
=&\cot(\phi)\mathscr{L}_{\mathrm{NS}}(Z_A)|_q+\nu(A\star F_X|_qX_{(\cdot)})|_q\\
&-\Gamma^1_{(1,2)}\nu(A\star E_2)|_q+\Gamma^1_{(1,2)}\cot(\phi)\nu(A\star X_A)|_q\\
=&\cot(\phi)F_Z|_q+\Gamma^1_{(1,2)}\cot(\phi)\nu(A\star X_A)|_q,\\
[F_Z,\nu((\cdot)\star X_{(\cdot)})]|_q =&-\mathscr{L}_{\mathrm{NS}}(\cot(\phi)X_A)|_q+\cot(\phi)\Gamma^1_{(1,2)}\nu(A\star Z_A)|_q\\
=&-\cot(\phi)F_X|_q.
\end{aligned}$$

Therefore, it remains to us to prove that the brackets of F_Y with all the other 7 generators of VF^2_Δ, as listed above, also belong to VF^2_Δ. Since the expression of F_Y involves λ, which was defined earlier, we need to know its derivatives in all the possible directions (except in F_Y-direction) as well as the expression for $F_Y|_q\theta$. We begin by computing this latter derivative. As usual, the way to proceed is to derivate $0=\hat{g}(AZ_A,\hat{E}_2)$ w.r.t. $F_Y|_q$, for which, we first compute

$$F_Y|_qZ_{(\cdot)}=(-F_Y|_q\theta+\Gamma^2_{(3,1)})X_A,$$

and hence (notice that $F_Y|_q\hat{E}_2=0$)

$$0=\hat{g}(-\lambda A(\star Z_A)Z_A,\hat{E}_2)+(-F_Y|_q\theta+\Gamma^2_{(3,1)})\hat{g}(AX_A,\hat{E}_2)=s_\phi(-F_Y|_q\theta+\Gamma^2_{(3,1)}),$$

from where one deduces that $F_Y|_q\theta = \Gamma^2_{(3,1)}$. One then easily computes that on $\tilde{O}_0$,

$$F_Y|_q X_{(\cdot)} = 0, \quad F_Y|_q E_2 = 0, \quad F_Y|_q Z_{(\cdot)} = 0.$$

To compute the derivatives of λ, we differentiate the identity $s_\phi\lambda = c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}$ proved above. Obviously, this will require the knowledge of derivatives of ϕ, so we begin there. To do that, one will differentiate the identity $c_\phi = \hat{g}(AE_2, \hat{E}_2)$ in different directions. One has,

$$\begin{aligned}
\hat{\nabla}_{AX_A}\hat{E}_2 &= -c_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A, \\
\hat{\nabla}_{AE_2}\hat{E}_2 &= s_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A, \\
\hat{\nabla}_{AZ_A}\hat{E}_2 &= -\hat{\Gamma}^1_{(1,2)}\hat{Z}_A,
\end{aligned}$$

and hence

$$\begin{aligned}
-s_\phi\mathscr{L}_{\mathrm{R}}(X_A)|_q\phi =& \hat{g}(-\Gamma^1_{(1,2)}AX_A, \hat{E}_2) + \hat{g}(AE_2, \hat{\nabla}_{AX_A}\hat{E}_2), \\
=& - s_\phi\Gamma^1_{(1,2)} + \hat{g}(AE_2, -c_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A), \\
=& - s_\phi\Gamma^1_{(1,2)} + s_\phi c_\phi\hat{\Gamma}^1_{(1,2)}, \\
-s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\phi =& \hat{g}(A\mathscr{L}_{\mathrm{R}}(E_2)|_q E_2, \hat{E}_2) + \hat{g}(AE_2, \hat{\nabla}_{AE_2}\hat{E}_2), \\
=& 0 + \hat{g}(AE_2, s_\phi\hat{\Gamma}^1_{(1,2)}\hat{X}_A) = -s_\phi^2\hat{\Gamma}^1_{(1,2)}, \\
-s_\phi\mathscr{L}_{\mathrm{R}}(Z_A)|_q\phi =& \hat{g}(-\Gamma^1_{(1,2)}AZ_A, \hat{E}_2) + \hat{g}(AE_2, -\hat{\Gamma}^1_{(1,2)}\hat{Z}_A) = 0, \\
-s_\phi\nu(A\star E_2)|_q\phi =& \hat{g}(A(\star E_2)E_2, \hat{E}_2) = 0, \\
-s_\phi\nu(A\star X_A)|_q\phi =& \hat{g}(A(\star X_A)E_2, \hat{E}_2) = \hat{g}(AZ_A, \hat{E}_2) = 0, \\
-s_\phi F_X|_q\phi =& \hat{g}(-\Gamma^1_{(1,2)}A(\star Z_A)E_2 - \Gamma^1_{(1,2)}AX_A, \hat{E}_2) = 0, \\
-s_\phi F_Z|_q\phi =& \hat{g}(-\Gamma^1_{(1,2)}AZ_A, \hat{E}_2) = 0, \\
-s_\phi F_Y|_q\phi =& \hat{g}(-\lambda A(\star Z_A)E_2 + 0, \hat{E}_2) = \lambda\hat{g}(AX_A, \hat{E}_2) = s_\phi\lambda.
\end{aligned}$$

Because $s_\phi \neq 0$ on $\tilde{O}_0$, one also gets

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q\phi =& \Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)}, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q\phi =& s_\phi\hat{\Gamma}^1_{(1,2)}, \\
F_Y|_q\phi =& -\lambda, \\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q\phi =& \nu(A\star E_2)|_q\phi = \nu(A\star X_A)|_q\phi = F_X|_q\phi = F_Z|_q\phi = 0.
\end{aligned}$$

Next notice that

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(X_A)|_q\Gamma^1_{(1,2)} &= F_X|_q\Gamma^1_{(1,2)} = X_A(\Gamma^1_{(1,2)}) = 0, \\
\mathscr{L}_{\mathrm{R}}(E_2)|_q\Gamma^1_{(1,2)} &= F_Y|_q\Gamma^1_{(1,2)} = E_2(\Gamma^1_{(1,2)}), \\
\mathscr{L}_{\mathrm{R}}(Z_A)|_q\Gamma^1_{(1,2)} &= F_Z|_q\Gamma^1_{(1,2)} = Z_A(\Gamma^1_{(1,2)}) = 0,
\end{aligned}$$

because $X_A, Z_A \in E_2^\perp$ and similarly, since $\hat{X}_A, \hat{Z}_A \in \hat{E}_2^\perp$,

$$\begin{aligned}
&\mathscr{L}_{\mathrm{R}}(X_A)|_q\hat{\Gamma}^1_{(1,2)} = AX_A(\hat{\Gamma}^1_{(1,2)}) = s_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}), \\
&\mathscr{L}_{\mathrm{R}}(E_2)|_q\hat{\Gamma}^1_{(1,2)} = AE_2(\hat{\Gamma}^1_{(1,2)}) = c_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}), \\
&\mathscr{L}_{\mathrm{R}}(Z_A)|_q\hat{\Gamma}^1_{(1,2)} = AZ_A(\hat{\Gamma}^1_{(1,2)}) = 0, \\
&F_X|_q\hat{\Gamma}^1_{(1,2)} = F_Y|_q\hat{\Gamma}^1_{(1,2)} = F_Z|_q\hat{\Gamma}^1_{(1,2)} = 0.
\end{aligned}$$

Finally, derivating the identity $s_\phi\lambda = c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}$ and using the previously derived rules,

$$
\begin{aligned}
&c_\phi(\Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)})\lambda + s_\phi\mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda = -s_\phi\Gamma^1_{(1,2)}(\Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)}) - s_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
&s_\phi c_\phi\hat{\Gamma}^1_{(1,2)}\lambda + s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda = -s_\phi^2\hat{\Gamma}^1_{(1,2)}\Gamma^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
&s_\phi\mathscr{L}_{\mathrm{R}}(Z_A)|_q\lambda = 0,\\
&s_\phi\nu(A\star E_2)|_q\lambda = 0,\\
&s_\phi\nu(A\star X_A)|_q\lambda = 0,\\
&s_\phi F_X|_q\lambda = 0,\\
&-c_\phi\lambda^2 + s_\phi F_Y|_q\lambda = s_\phi\Gamma^1_{(1,2)}\lambda + c_\phi E_2(\Gamma^1_{(1,2)}),\\
&s_\phi F_Z|_q\lambda = 0,
\end{aligned}
$$

from which the last 6 simplify immediately to

$$
\begin{aligned}
\mathscr{L}_{\mathrm{R}}(Z_A)|_q\lambda =&\nu(A\star E_2)|_q\lambda = \nu(A\star X_A)|_q\lambda = F_X|_q\lambda = F_Z|_q\lambda = 0,\\
F_Y|_q\lambda =&\cot(\phi)(E_2(\Gamma^1_{(1,2)}) + \lambda^2) + \Gamma^1_{(1,2)}\lambda.
\end{aligned}
$$

Next simplify $\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda$ by using first $s_\phi\lambda = c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}$, and obtain

$$
\begin{aligned}
s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda =& - s_\phi c_\phi\hat{\Gamma}^1_{(1,2)}\lambda - s_\phi^2\hat{\Gamma}^1_{(1,2)}\Gamma^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
=& - c_\phi\hat{\Gamma}^1_{(1,2)}(c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}) - s_\phi^2\hat{\Gamma}^1_{(1,2)}\Gamma^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
=& - \Gamma^1_{(1,2)}\hat{\Gamma}^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) + c_\phi(-\hat{E}_2(\hat{\Gamma}^1_{(1,2)}) + (\hat{\Gamma}^1_{(1,2)})^2)
\end{aligned}
$$

and then using $-K = -\hat{E}_2(\hat{\Gamma}^1_{(1,2)}) + (\hat{\Gamma}^1_{(1,2)})^2$, to deduce

$$
s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda = - \Gamma^1_{(1,2)}\hat{\Gamma}^1_{(1,2)} + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi K,
$$

once more $\hat{\Gamma}^1_{(1,2)} = c_\phi\Gamma^1_{(1,2)} - s_\phi\lambda$,

$$
\begin{aligned}
s_\phi\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda =& - \Gamma^1_{(1,2)}(c_\phi\Gamma^1_{(1,2)} - s_\phi\lambda) + c_\phi E_2(\Gamma^1_{(1,2)}) - c_\phi K,\\
=&c_\phi(-K - (\Gamma^1_{(1,2)})^2 + E_2(\Gamma^1_{(1,2)})) + s_\phi\Gamma^1_{(1,2)}\lambda,
\end{aligned}
$$

which finally simplifies, thanks to $-K = -E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2$ and $s_\phi \neq 0$, to

$$
\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda = \lambda\Gamma^1_{(1,2)}.
$$

Next we simplify $\mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda$ by using the same identities as above when simplifying $\mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda$ yields

$$
\begin{aligned}
s_\phi\mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda =& - c_\phi(\Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)})\lambda - s_\phi\Gamma^1_{(1,2)}(\Gamma^1_{(1,2)} - c_\phi\hat{\Gamma}^1_{(1,2)}) - s_\phi\hat{E}_2(\hat{\Gamma}^1_{(1,2)}),\\
=& - \lambda(s_\phi\lambda + \hat{\Gamma}^1_{(1,2)}) + c_\phi^2\hat{\Gamma}^1_{(1,2)}\lambda,\\
&- s_\phi(\Gamma^1_{(1,2)})^2 + s_\phi\hat{\Gamma}^1_{(1,2)}(s_\phi\lambda + \hat{\Gamma}^1_{(1,2)}) - s_\phi(K + (\hat{\Gamma}^1_{(1,2)})^2),\\
=& - s_\phi(\lambda^2 + (\Gamma^1_{(1,2)})^2 + K) - \lambda\hat{\Gamma}^1_{(1,2)} + c_\phi^2\lambda\hat{\Gamma}^1_{(1,2)} + s_\phi^2\hat{\Gamma}^1_{(1,2)}\lambda,\\
=& - s_\phi(\lambda^2 + (\Gamma^1_{(1,2)})^2 + K),
\end{aligned}
$$

which implies, at last, that $\mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda = -(\lambda^2 + (\Gamma^1_{(1,2)})^2 + K)$.

Finally, on $\tilde{O}_0$, we compute Lie the brackets

$$\begin{aligned}
[\mathscr{L}_{\mathrm{R}}(X_A), F_Y]|_q =& \mathscr{L}_{\mathrm{NS}}(-\Gamma^1_{(1,2)}X_A)|_q - \mathscr{L}_{\mathrm{R}}(\mathscr{L}_{\mathrm{NS}}(E_2)|_q X_{(\cdot)})|_q \\
& + \nu(AR(X_A \wedge E_2) - \hat{R}(AX_A \wedge 0)A)|_q - \mathscr{L}_{\mathrm{R}}(X_A)|_q \lambda\nu(A \star Z_A)|_q \\
& - \lambda\big(- \mathscr{L}_{\mathrm{NS}}(A(\star Z_A)X_A) - \mathscr{L}_{\mathrm{R}}(\nu(A\star Z_A)|_q X_{(\cdot)}) + \nu(A\star 0)|_q\big) \\
=& - \Gamma^1_{(1,2)}F_X|_q - \mathscr{L}_{\mathrm{R}}(F_Y|_q X_{(\cdot)})|_q + \lambda\mathscr{L}_{\mathrm{R}}(E_2)|_q - \lambda F_Y|_q \\
& + \underbrace{(-(\Gamma^1_{(1,2)})^2 - K - \mathscr{L}_{\mathrm{R}}(X_A)|_q\lambda - \lambda^2)}_{=0}\nu(A\star Z_A)|_q, \\
[\mathscr{L}_{\mathrm{R}}(E_2), F_Y]|_q =& - \mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda\nu(A\star Z_A)|_q - \lambda(-\mathscr{L}_{\mathrm{NS}}(A(\star Z_A)E_2)|_q + \nu(A\star 0)|_q) \\
=& - \lambda\mathscr{L}_{\mathrm{R}}(X_A)|_q + \lambda F_X|_q + \underbrace{(\lambda\Gamma^1_{(1,2)} - \mathscr{L}_{\mathrm{R}}(E_2)|_q\lambda)}_{=0}\nu(A\star Z_A)|_q, \\
[\mathscr{L}_{\mathrm{R}}(Z_A), F_Y]|_q =& \mathscr{L}_{\mathrm{NS}}(-\Gamma^1_{(1,2)}Z_A)|_q + \mathscr{L}_{\mathrm{R}}(\mathscr{L}_{\mathrm{NS}}(E_2)|_q Z_{(\cdot)}) \\
& + \nu(AR(Z_A\wedge E_2) - \hat{R}(AZ_A\wedge 0)A)|_q - \mathscr{L}_{\mathrm{R}}(Z_A)|_q\lambda\nu(A\star Z_A)|_q \\
& - \lambda\big(- \mathscr{L}_{\mathrm{NS}}(A(\star Z_A)Z_A)|_q + \mathscr{L}_{\mathrm{R}}(\nu(A\star Z_A)|_q Z_{(\cdot)})|_q) \\
& - \lambda\nu(A\star(-\lambda X_A + \Gamma^1_{(1,2)}E_2)|_q \\
=& - \Gamma^1_{(1,2)}F_Z|_q + \mathscr{L}_{\mathrm{R}}(F_Y|_q Z_{(\cdot)})|_q + K\nu(A\star X_A)|_q \\
& - \underbrace{\mathscr{L}_{\mathrm{R}}(Z_A)|_q\lambda}_{=0}\nu(A\star Z_A)|_q - \lambda\nu(A\star(-\lambda X_A + \Gamma^1_{(1,2)}E_2)|_q,
\end{aligned}$$

$$\begin{aligned}
[\nu((\cdot)\star E_2), F_Y]|_q =& - \nu(A\star E_2)|_q\lambda\nu(A\star Z_A)|_q \\
& - \lambda\nu(A[\star E_2, \star Z_A]_{\mathfrak{so}} - \nu(A\star E_2)|_q\theta A\star X_A)|_q \\
=& - \nu(A\star E_2)|_q\lambda\nu(A\star Z_A)|_q = 0, \\
[\nu((\cdot)\star X_{(\cdot)}, F_Y]|_q =& - \nu(A\star\mathscr{L}_{\mathrm{NS}}(E_2)|_q X_{(\cdot)})|_q - \nu(A\star X_A)|_q\lambda\nu(A\star Z_A)|_q \\
& - \lambda\nu(A[\star X_A, \star Z_A]_{\mathfrak{so}} - \nu(A\star X_A)|_q\theta A\star X_A)|_q \\
& - \lambda\nu(-A\star\nu(A\star Z_A)|_q X_{(\cdot)}))|_q \\
=& - \nu(A\star\underbrace{F_Y|_q X_{(\cdot)}}_{=0})|_q - \underbrace{\nu(A\star X_A)|_q\lambda}_{=0}\nu(A\star Z_A)|_q \\
& - \lambda\nu(A\star(-E_2 + \cot(\phi)X_A))|_q, \\
[F_Z, F_Y]|_q =& \mathscr{L}_{\mathrm{NS}}(-\Gamma^1_{(1,2)}Z_A - \mathscr{L}_{\mathrm{NS}}(E_2)|_q Z_{(\cdot)})|_q + \nu(AR(Z_A\wedge E_2))|_q \\
& - F_Z|_q\lambda\nu(A\star Z_A)|_q - \lambda(-\mathscr{L}_{\mathrm{NS}}(\nu(A\star Z_A)|_q Z_{(\cdot)}) + \nu(A\star F_Z|_q Z_{(\cdot)})|_q) \\
=& - \Gamma^1_{(1,2)}F_Z|_q - \mathscr{L}_{\mathrm{NS}}(\underbrace{F_Y|_q Z_{(\cdot)}}_{=0})|_q + K\nu(A\star X_A)|_q \\
& - \underbrace{F_Z|_q\lambda}_{=0}\nu(A\star Z_A)|_q - \lambda\nu(A\star(-\cot(\phi)X_A + \Gamma^1_{(1,2)}E_2))|_q,
\end{aligned}$$

and finally, noticing that $-\lambda F_X|_q + \Gamma^1_{(1,2)} F_Y|_q = -\lambda \mathscr{L}_{\rm NS}(X_A)|_q + \Gamma^1_{(1,2)}\mathscr{L}_{\rm NS}(E_2)|_q$,

$$\begin{aligned}
[F_X, F_Y]|_q =& \mathscr{L}_{\rm NS}(\mathscr{L}_{\rm NS}(X_A)|_q E_2 - \mathscr{L}_{\rm NS}(E_2)|_q X_{(\cdot)})|_q + \nu(AR(X_A \wedge E_2))|_q \\
& - \mathscr{L}_{\rm R}(X_A)|_q \lambda \nu(A \star Z_A)|_q + E_2(\Gamma^1_{(1,2)})\nu(A\star Z_A)|_q \\
& - \lambda(-\mathscr{L}_{\rm NS}(\nu(A\star Z_A)|_q X_{(\cdot)}) + \nu(A \star \mathscr{L}_{\rm NS}(X_A)|_q Z_{(\cdot)})|_q) \\
& + \Gamma^1_{(1,2)}\nu(A\star \mathscr{L}_{\rm NS}(E_2)|_q Z_{(\cdot)})|_q + \Gamma^1_{(1,2)}\nu(A\star Z_A)|_q \lambda \nu(A\star Z_A)|_q \\
=& -\Gamma^1_{(1,2)}\mathscr{L}_{\rm NS}(X_A)|_q - \mathscr{L}_{\rm NS}(\underbrace{F_Y|_q X_{(\cdot)}}_{=0})|_q \\
& + \nu(A \star \underbrace{(-\lambda F_X|_q + \Gamma^1_{(1,2)} F_Y|_q) Z_{(\cdot)}}_{=0})|_q \\
& + (-K - F_X|_q \lambda + E_2(\Gamma^1_{(1,2)}))\nu(A\star Z_A)|_q \\
=& -\Gamma^1_{(1,2)} F_X|_q + (-K - F_X|_q\lambda + E_2(\Gamma^1_{(1,2)}) - (\Gamma^1_{(1,2)})^2)\nu(A\star Z_A)|_q,
\end{aligned}$$

which, after using $F_X|_q\lambda = 0$ and Eq. (57), simplifies to $[F_X, F_Y]|_q = -\Gamma^1_{(1,2)} F_X|_q$. Since all these Lie brackets also belong to VF^2_Δ, we conclude that VF^2_Δ is involutive and therefore $\mathrm{Lie}(\Delta) = \mathrm{VF}^2_\Delta$. Therefore the span of $\mathrm{Lie}(\Delta)$ at each point $\tilde{O}_0$ is 8-dimensional subspace of $T|_q Q$, since VF^2_Δ is generated by 8 pointwise linearly independent vector fields. Since $q_0 \in Q\backslash S_1$ was arbitrary and since the choice of X_A, E_2, Z_A in $\tilde{O}_0$ are unique up to multiplication by -1, we have shown that on Q_0 there is a smooth 5-dimensional distribution Δ containing $\mathcal{D}_{\rm R}|_{Q_0}$ such that $\mathrm{Lie}(\Delta) = \mathrm{VF}^2_\Delta$ spans an 8-dimensional distribution $\mathcal{D}$ and which is then, by construction, involutive. We already know from the beginning of the proof that $q \in S_1$ implies that $\mathcal{O}_{\mathcal{D}_{\rm R}}(q) \subset S_1$ so, equivalently, $q \in Q\backslash S_1$ implies that $\mathcal{O}_{\mathcal{D}_{\rm R}}(q) \subset Q\backslash S_1$. Hence we are interested to see how $\mathcal{D}$ can be extended on all over $Q\backslash S_1$ i.e. we have to see how to define it on $Q_1\backslash S_1$. For this purpose, we define the Sasaki metric G on Q by

$$\begin{aligned}
&\mathcal{X} = \mathscr{L}_{\rm NS}(X, \hat{X})|_q + \nu(A\star Z)|_q, \quad \mathcal{Y} = \mathscr{L}_{\rm NS}(Y, \hat{Y})|_q + \nu(A\star W)|_q, \\
&G(\mathcal{X}, \mathcal{Y}) = g(X, Y) + \hat{g}(\hat{X}, \hat{Y}) + g(Z, W),
\end{aligned}$$

for $q = (x, \hat{x}; A) \in Q$, $X, Y, Z, W \in T|_x M$, $\hat{X}, \hat{Y} \in T|_{\hat{x}}\hat{M}$. Notice that any vector $\mathcal{X} \in T|_q Q$ can be written in the form $\mathscr{L}_{\rm NS}(X, \hat{X})|_q + \nu(A\star Z)|_q$ for some $X, \hat{X}, Z$ as above. Since $\mathcal{D}$ is a smooth codimension 1 distribution on Q_0, it has a smooth normal line bundle $\mathcal{D}^\perp$ w.r.t. G defined on Q_0 which uniquely determines $\mathcal{D}$. We will use the Sasaki metric G to determine a smooth vector field $\mathcal{N}$ near a point $q_0 \in Q_1\backslash S_1$ spanning $\mathcal{D}^\perp$. So let $q_0 \in Q_1\backslash S_1$ and assume, as before, that $q_0 \in Q_1^+\backslash S_1$ the case of $Q_1^-\backslash S_1$ being handled similarly. Take the frames E_1, E_2, E_3, $\hat{E}_1, \hat{E}_2, \hat{E}_3$ and $\tilde{O}$, $\tilde{O}_0$, X_A, Z_A as done above (the case (b)). Because $\cos(\phi(q_0))\Gamma^1_{(1,2)}(x_0) - \hat{\Gamma}^1_{(1,2)}(\hat{x}_0) \neq 0$, one assumes, after shrinking $\tilde{O}$ around q_0, that we have $\cos(\phi(q))\Gamma^1_{(1,2)}(x) - \hat{\Gamma}^1_{(1,2)}(\hat{x}) \neq 0$ for all $q = (x, \hat{x}; A) \in \tilde{O}$, which then implies that $\lambda(q) \neq 0$ on $\tilde{O}_0$. Here to say what is the value of $\cos(\phi(q))$ even at $q \in Q_1\backslash S_1$, we use the fact that $\cos(\phi(q)) = g(AE_2, \hat{E}_2)$ for all $q \in \tilde{O}$ (though $\phi(q)$ is not *a priori* defined). To determine a smooth vector field $\mathcal{N} \in \mathcal{D}^\perp$ on $\tilde{O}_0$, we write

$$\begin{aligned}
\mathcal{N}|_q =& a_1\mathscr{L}_{\rm NS}(X_A)|_q + a_2 + \mathscr{L}_{\rm NS}(E_2)|_q + a_3\mathscr{L}_{\rm NS}(Z_A)|_q \\
& + b_1\mathscr{L}_{\rm NS}(AX_A)|_q + b_2 + \mathscr{L}_{\rm NS}(AE_2)|_q + b_3\mathscr{L}_{\rm NS}(AZ_A)|_q \\
& + v_1\nu(A\star X_A)|_q + v_2\nu(A\star E_2)|_q + v_3\nu(A\star Z_A)|_q,
\end{aligned}$$

and since this must be G-orthogonal to $\mathcal{D}$, we get

$$
\begin{aligned}
0 =& G(\mathcal{N}, \mathscr{L}_{\mathrm{R}}(X_A)) = a_1 + b_1, \quad 0 = G(\mathcal{N}, \mathscr{L}_{\mathrm{R}}(E_2)) = a_2 + b_2, \\
0 =& G(\mathcal{N}, \mathscr{L}_{\mathrm{R}}(Z_A)) = a_3 + b_3, \ 0 = G(\mathcal{N}, \nu(A \star X_A)) = v_1, \ 0 = G(\mathcal{N}, \nu(A \star E_2)) = v_2, \\
0 =& G(\mathcal{N}, F_X) = a_1 - \Gamma^1_{(1,2)} v_3, \quad 0 = G(\mathcal{N}, F_Y) = a_2 - \lambda v_3, \quad 0 = G(\mathcal{N}, F_Z) = a_3.
\end{aligned}
$$

So if we set $v_3 = \frac{1}{\lambda}$ and introduce the notation

$$
\mathscr{L}^{\perp}_{\mathrm{R}}(X)|_q := \mathscr{L}_{\mathrm{NS}}(X, -AX) \in \mathcal{D}_{\mathrm{NS}}|_q, \quad q = (x, \hat{x}; A) \in Q, \ X \in T|_x M,
$$

we get a smooth vector field $\mathcal{N}$ on $\tilde{O}_0$ which is G-perpendicular to $\mathcal{D}$ and given by

$$
\begin{aligned}
\mathcal{N}|_q =& \frac{1}{\lambda(q)} \Gamma^1_{(1,2)}(x) \mathscr{L}^{\perp}_{\mathrm{R}}(X_A)|_q + \mathscr{L}^{\perp}_{\mathrm{R}}(E_2) + \frac{1}{\lambda(q)} \nu(A \star Z_A)|_q, \quad q \in \tilde{O}_0 \\
=& \frac{c_\theta}{\lambda(q)} (\Gamma^1_{(1,2)} \mathscr{L}^{\perp}_{\mathrm{R}}(E_1)|_q + \nu(A \star E_3)|_q) + \mathscr{L}^{\perp}_{\mathrm{R}}(E_2)|_q \\
&+ \frac{s_\theta}{\lambda(q)} (\Gamma^1_{(1,2)} \mathscr{L}^{\perp}_{\mathrm{R}}(E_3)|_q - \nu(A \star E_1)|_q).
\end{aligned}
$$

i.e.,

$$
\mathcal{N}|_q = H_1(q) \mathcal{X}_1|_q + \mathcal{X}_2|_q + H_3(q) \mathcal{X}_3|_q,
$$

where $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ are pointwise linearly independent smooth vector fields on $\tilde{O}$ (and not only $\tilde{O}_0$) given by

$$
\begin{aligned}
\mathcal{X}_1|_q =& \Gamma^1_{(1,2)} \mathscr{L}^{\perp}_{\mathrm{R}}(E_1)|_q + \nu(A \star E_3)|_q, \\
\mathcal{X}_2|_q =& \mathscr{L}^{\perp}_{\mathrm{R}}(E_2)|_q, \\
\mathcal{X}_3|_q =& \Gamma^1_{(1,2)} \mathscr{L}^{\perp}_{\mathrm{R}}(E_3)|_q - \nu(A \star E_1)|_q,
\end{aligned}
$$

while H_1, H_3 are smooth functions on $\tilde{O}_0$ defined by

$$
H_1 = \frac{\cos(\theta)}{\lambda}, \quad H_3 = \frac{\sin(\theta)}{\lambda}.
$$

Notice that θ and λ cannot be extended in a smooth or even C^1-way from $\tilde{O}_0$ to $\tilde{O}$, but as we will show, one can extend H_1, H_3 in at least C^1-way onto $\tilde{O}$. First, since $\lambda(q) \to \pm\infty$ while $\cos(\theta(q))$, $\sin(\theta(q))$ stay bounded, it follows that H_1, H_3 extend uniquely to $\tilde{O} \cap Q_1$ by declaring $H_1(q) = H_3(q) = 0$ for all $q \in \tilde{O} \cap Q_1$. Of course, these extensions, which we still denote by H_1, H_3, are continuous functions on $\tilde{O}$.

The next objective consists of showing that H_1, H_3 are at least C^1 on $\tilde{O}$. For this purpose, let $\mathcal{X} \in \mathrm{VF}(\tilde{O})$ and decompose it uniquely as

$$
\mathcal{X} = \sum_{i=1}^{3} a_i \mathscr{L}_{\mathrm{R}}(E_i) + \sum_{i=1}^{3} b_i \mathscr{L}_{\mathrm{NS}}(E_i) + \sum_{i=1}^{3} v_i \nu((\cdot) \star E_i),
$$

with $a_i, b_i, v_i \in C^\infty(\tilde{O})$. We will need to know the derivatives of θ and λ in all the directions on $\tilde{O}_0$. These have been computed above by using the frame X_A, E_2, Z_A

instead of E_1, E_2, E_3 except in the direction of $\nu(A\star Z_A)|_q$. As before, one computes (using that $s_\phi \neq 0$ on $\tilde{O}_0$ as usual),

$$
\begin{aligned}
&\nu(A\star Z_A)|_q\theta = 0, \quad \nu(A\star Z_A)|_q\phi = 1, \\
&\nu(A\star Z_A)|_q\lambda = -\Gamma^1_{(1,2)}(x) - \lambda(q)\cot(\phi(q)).
\end{aligned}
$$

One now easily computes that on $\tilde{O}_0$,

$$
\begin{aligned}
\mathcal{X}(\theta) =&(-a_1 s_\theta + a_3 c_\theta)\lambda + (-b_1 s_\theta \Gamma^1_{(1,2)} + b_3 c_\theta \Gamma^1_{(1,2)} - v_1 c_\theta - v_3 s_\theta)\cot(\phi) + B_1(q), \\
\mathcal{X}(\lambda) =&(-a_1 c_\theta - a_3 s_\theta)\lambda^2 + (-b_1 c_\theta \Gamma^1_{(1,2)} - b_3 s_\theta \Gamma^1_{(1,2)} + v_1 s_\theta - v_3 c_\theta)\lambda\cot(\phi), \\
&+ a_2\Gamma^1_{(1,2)}\lambda + b_2\cot(\phi)E_2(\Gamma^1_{(1,2)}) + B_2(q),
\end{aligned}
$$

where

$$
\begin{aligned}
B_1(q) &= (a_1+b_1)\Gamma^1_{(3,1)} + (a_2+b_2)\Gamma^2_{(3,1)} + (a_3+b_3)\Gamma^3_{(3,1)} + v_2, \\
B_2(q) &= (-a_1 c_\theta - a_3 s_\theta)((\Gamma^1_{(1,2)})^2 + K) + (-b_1 c_\theta - b_3 s_\theta)(\Gamma^1_{(1,2)})^2 + (v_1 s_\theta - v_3 c_\theta)\Gamma^1_{(1,2)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{X}(H_1) =& - s_\theta\frac{\mathcal{X}(\theta)}{\lambda} - c_\theta\frac{\mathcal{X}(\lambda)}{\lambda^2} \\
=&a_1 + (b_1\Gamma^1_{(1,2)} + v_3)\frac{\cot(\phi)}{\lambda} - \frac{a_2 c_\theta\Gamma^1_{(1,2)}}{\lambda} - \frac{b_2 c_\theta E_2(\Gamma^1_{(1,2)})}{\lambda}\frac{\cot(\phi)}{\lambda} - \frac{s_\theta B_1}{\lambda} - \frac{c_\theta B_2}{\lambda^2}, \\
\mathcal{X}(H_3) =&c_\theta\frac{\mathcal{X}(\theta)}{\lambda} - s_\theta\frac{\mathcal{X}(\lambda)}{\lambda^2} \\
=&a_3 + (b_3\Gamma^1_{(1,2)} - v_1)\frac{\cot(\phi)}{\lambda} - \frac{a_2 s_\theta\Gamma^1_{(1,2)}}{\lambda} - \frac{b_3 s_\theta E_2(\Gamma^1_{(1,2)})}{\lambda}\frac{\cot(\phi)}{\lambda} + \frac{c_\theta B_1}{\lambda} - \frac{s_\theta B_2}{\lambda^2}.
\end{aligned}
$$

Since $s_\phi\lambda = c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}$, one has

$$
\frac{\cot(\phi)}{\lambda} = \frac{c_\phi}{c_\phi\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}},
$$

and therefore as q tends to a point q_1 of $Q_1^+ \cap \tilde{O}$, we have

$$
\lim_{q\to q_1}\frac{\cot(\phi)}{\lambda} = \frac{1}{\Gamma^1_{(1,2)} - \hat{\Gamma}^1_{(1,2)}}.
$$

Since B_1, B_2 stay bounded as q approaches a point of $Q_1^+ \cap \tilde{O}$, we get for every $q_1 = (x_1, \hat{x}_1; A_1) \in Q_1^+ \cap \tilde{O}$ that

$$
\begin{aligned}
\lim_{q\to q_1}\mathcal{X}(H_1) = a_1(q_1) + \frac{b_1(q_1)\Gamma^1_{(1,2)}(x_1) + v_3(q_1)}{\Gamma^1_{(1,2)}(x_1) - \hat{\Gamma}^1_{(1,2)}(\hat{x}_1)} =: D_{\mathcal{X}}H_1(q_1), \\
\lim_{q\to q_1}\mathcal{X}(H_3) = a_3(q_1) + \frac{b_3(q_1)\Gamma^1_{(1,2)}(x_1) - v_1(q_1)}{\Gamma^1_{(1,2)}(x_1) - \hat{\Gamma}^1_{(1,2)}(\hat{x}_1)} =: D_{\mathcal{X}}H_3(q_1).
\end{aligned}
$$

From these, it is now readily seen that H_1, H_3 are differentiable on $\tilde{O} \cap Q_1^+$ with $\mathcal{X}|_{q_1}(H_1) = D_{\mathcal{X}}H_1(q_1)$, $\mathcal{X}|_{q_1}(H_3) = D_{\mathcal{X}}H_3(q_1)$ and that H_1, H_3 are C^1-functions on

$\tilde{O}$. We therefore have that $\mathcal{N}$ is a well-defined C^1 vector field on $\tilde{O}$ and since $\mathcal{D}=\mathcal{N}^\perp$ w.r.t. G on $\tilde{O}_0$, it follows that $\mathcal{D}$ extends in C^1-sense on $\tilde{O}$. Since $q_0\in Q_1^+\backslash S_1$ was arbitrary and because the case $q_0\in Q_1^-\backslash S_1$ is handled similarly, we see that $\mathcal{D}$ can be extended onto the open subset $Q\backslash S_1$ of Q as a (at least) C^1-distribution, which is C^∞ on Q_0. Since $\mathcal{D}_{\mathrm{R}}|_{Q\backslash S_1}\subset\mathcal{D}$ and because $q\in Q\backslash S_1$ implies that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)\subset Q\backslash S_1$ as we have seen, it follows that for every $q_0\in Q\backslash S_1$ we have $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset\mathcal{O}_{\mathcal{D}}(q_0)$ where the orbit on the right is *a priori* an immersed C^1-submanifold of $Q\backslash S_1$. However, since $\mathcal{D}$ is involutive and $\dim\mathcal{D}=8$ on $Q\backslash S_1$, we get by the C^1-version of the Frobenius theorem that $\dim\mathcal{O}_{\mathcal{D}}(q_0)=8$ and hence

$$\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\leq\dim\mathcal{O}_{\mathcal{D}}(q_0)=8,$$

for every $q_0\in Q\backslash S_1$.

We will now investigate when the equality holds here. Define

$$\begin{aligned}
M_0&=\{x\in M\mid K_2(x)\neq K\},\\
M_1&=\{x\in M\mid\exists\text{ open }V\ni x\text{ s.t. }K_2(x')=K\ \forall x'\in V\},\\
\hat{M}_0&=\{\hat{x}\in\hat{M}\mid\hat{K}_2(\hat{x})\neq K\},\\
\hat{M}_1&=\{\hat{x}\in\hat{M}\mid\exists\text{ open }\hat{V}\ni\hat{x}\text{ s.t. }K_2(\hat{x}')=K\ \forall\hat{x}'\in\hat{V}\},
\end{aligned}$$

and notice that $M_0\cup M_1$ (resp. $\hat{M}_0\cup\hat{M}_1$) is a dense subset of M (resp. $\hat{M}$). Here we also fix the choice of $q_0=(x_0,\hat{x}_0;A_0)\in Q\backslash S_1$ and define $M^\circ=\pi_{Q,M}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$, $\hat{M}^\circ=\pi_{Q,\hat{M}}(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$ as in the statement. Write also $Q^\circ:=\pi_Q^{-1}(M^\circ\times\hat{M}^\circ)$ and notice that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset Q^\circ$. We define on Q two 2-dimensional distributions D and $\hat{D}$. For every $q_1=(x_1,\hat{x}_1;A_1)\in Q$, take orthonormal frames E_1,E_2,E_3, $\hat{E}_1,\hat{E}_2,\hat{E}_3$ of $M,\hat{M}$ defined on open neighbourhoods $U,\hat{U}$ of $x_1,\hat{x}_1$ with $E_2=\frac{\partial}{\partial r}$, $\hat{E}_2=\frac{\partial}{\partial r}$. Then, for $q\in\pi_Q^{-1}(U\times\hat{U})\cap Q$, the 2-dimensional plane $D|_q$ is spanned by

$$\begin{aligned}
K_1|_q&=\mathscr{L}_{\mathrm{NS}}(A^{\overline{T}}\hat{E}_1)|_q-\hat{\Gamma}^1_{(1,2)}(x)\nu((\hat{\star}\hat{E}_3)A)|_q,\\
K_3|_q&=\mathscr{L}_{\mathrm{NS}}(A^{\overline{T}}\hat{E}_3)|_q+\hat{\Gamma}^1_{(1,2)}(x)\nu((\hat{\star}\hat{E}_1)A)|_q,
\end{aligned}$$

and $\hat{D}|_q$ is spanned by

$$\begin{aligned}
\hat{K}_1|_q&=\mathscr{L}_{\mathrm{NS}}(AE_1)|_q+\Gamma^1_{(1,2)}(x)\nu(A\star E_3)|_q,\\
\hat{K}_3|_q&=\mathscr{L}_{\mathrm{NS}}(AE_3)|_q-\Gamma^1_{(1,2)}(x)\nu(A\star E_1)|_q.
\end{aligned}$$

Obviously, different choices of frames $E_i,\hat{E}_i$, $i=1,2,3$, give $K_1,K_3,\hat{K}_1,\hat{K}_3$ that span the same planes $D,\hat{D}$, since we have fixed the choice of $E_2=\frac{\partial}{\partial r}$, $\hat{E}_2=\frac{\partial}{\partial r}$. Exactly as in proof of Proposition 5.31, one can show that for every $q_1=((r_1,y_1),(\hat{r}_1,\hat{y}_1);A_1)\in Q$ and smooth paths $\gamma:[0,1]\to N$, $\hat{\gamma}:[0,1]\to\hat{N}$ with $\gamma(0)=y_1$, $\hat{\gamma}(0)=\hat{y}_1$ there are unique smooth paths $\Gamma,\hat{\Gamma}:[0,1]\to Q$ such that for all $t\in[0,1]$,

$$\begin{aligned}
&\dot{\Gamma}(t)\in D|_{\Gamma(t)},\quad\Gamma(0)=q_1,\quad(\pi_{Q,M}\circ\Gamma)(t)=(r_1,\gamma(t)),\\
&\dot{\hat{\Gamma}}(t)\in\hat{D}|_{\hat{\Gamma}(t)},\quad\hat{\Gamma}(0)=q_1,\quad(\pi_{Q,\hat{M}}\circ\hat{\Gamma})(t)=(\hat{r}_1,\hat{\gamma}(t)).
\end{aligned}$$

Since $(\pi_{Q,\hat{M}})_*D=0$ (resp. $(\pi_{Q,M})_*\hat{D}=0$), one has $\pi_{Q,\hat{M}}(\Gamma(t))=\hat{x}_1$ (resp. $\pi_{Q,M}(\hat{\Gamma}(t))=x_1$) for all $t\in[0,1]$. We write these as $\Gamma=\Gamma(\gamma,q_1)$, $\hat{\Gamma}=\hat{\Gamma}(\hat{\gamma},q_1)$. If

$E_2 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{\partial}{\partial r}$, then by exactly the same arguments as in the proof of Proposition 5.31 we have

$$\begin{aligned}
&\nu(A \star E_2)|_q \in T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M_0 \times \hat{M}), \\
&\nu((\hat{\star}\hat{E}_2)A)|_q \in T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in Q_0 \cap \pi_Q^{-1}(M \times \hat{M}_0).
\end{aligned}$$

We next show how one can replace Q_0 by $Q \backslash S_1$. Take frames E_i, $\hat{E}_i$, $i = 1, 2, 3$, as above when defining $D, \hat{D}$ for some $q_1 \in Q_1 \backslash S_1$. We assume here without loss of generality that $q_1 \in Q_1^+ \backslash S_1$ since the case $q_1 \in Q_1^- \backslash S_1$ can be dealt with in a similar way. If $h_1, h_2 : \pi_Q^{-1}(U \times \hat{U}) \to \mathbb{R}$ are defined as $h_1(q) = \hat{g}(AE_1, \hat{E}_2)$, $h_2(q) = \hat{g}(AE_3, \hat{E}_2)$, we have $Q_1 \cap \pi_Q^{-1}(U \times \hat{U}) = (h_1, h_2)^{-1}(0)$ and $(h_1, h_2) : \pi_Q^{-1}(U \times \hat{U}) \to \mathbb{R}^2$ is a regular map at the points of Q_1 (see e.g. Remark 5.35 or the proof of Proposition 5.29). Since $q_1 \in Q_1^+ \backslash S_1$, then $\mathscr{L}_{\mathrm{R}}(E_1)|_{q_1} h_1 = \Gamma^1_{(1,2)}(x_1) - \hat{\Gamma}^1_{(1,2)}(\hat{x}_1) \neq 0$ and $\mathscr{L}_{\mathrm{R}}(E_3)|_{q_1} h_2 = \Gamma^1_{(1,2)}(x_1) - \hat{\Gamma}^1_{(1,2)}(\hat{x}_1) \neq 0$, which shows that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$ intersects Q_1^+ transversally at q_1 (hence at every point $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$), by dimensional reasons (because $\dim Q_1 = 7$, $\dim Q = 9$). From this, it follows that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1) \cap Q_1$ is a smooth closed submanifold of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$ and that there is a sequence $q'_n = (x'_n, \hat{x}'_n; A'_n) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1) \cap Q_0$ such that $q'_n \to q_1$. If now $q_1 \in \pi_Q^{-1}(M_0 \times \hat{M}) \cap Q_1 \backslash S_1$, then we know that for n large enough, $q'_n \in \pi_Q^{-1}(M_0 \times \hat{M}) \cap Q_0$ and hence $\nu(A \star E_2)|_{q'_n} \in T|_{q'_n} \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q'_n) = T|_{q'_n} \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Taking the limit implies that $\nu(A \star E_2)|_{q_1} \in T|_{q_1} \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Similarly, if $q_1 \in \pi_Q^{-1}(M \times \hat{M}_0) \cap Q_1 \backslash S_1$, one has $\nu((\hat{\star}\hat{E}_2)A)|_{q_1} \in T|_{q_1} \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)$. Hence we have that if $E_2 = \frac{\partial}{\partial r}$, $\hat{E}_2 = \frac{\partial}{\partial r}$, then

$$\begin{aligned}
&\nu(A \star E_2)|_q \in T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in (Q \backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M}), \\
&\nu((\hat{\star}\hat{E}_2)A)|_q \in T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in (Q \backslash S_1) \cap \pi_Q^{-1}(M \times \hat{M}_0).
\end{aligned}$$

For every $q \in (Q \backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M})$, which is an open subset of Q, one has $\nu(A \star E_2)|_q \in T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ with $E_2 = \frac{\partial}{\partial r}$ and hence by Proposition 5.5, case (i), it follows that

$$\begin{aligned}
L_1|_q &= \mathscr{L}_{\mathrm{NS}}(E_1)|_q - \Gamma^1_{(1,2)}(x)\nu(A \star E_3)|_q, \\
L_3|_q &= \mathscr{L}_{\mathrm{NS}}(E_3)|_q + \Gamma^1_{(1,2)}(x)\nu(A \star E_1)|_q,
\end{aligned}$$

are tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$, where $E_1, E_2 = \frac{\partial}{\partial r}, E_3$ is an orthonormal frame in an open neighbourhood of x_1. But because $\hat{K}_1|_q = \mathscr{L}_{\mathrm{R}}(E_1)|_q - L_1|_q$, $\hat{K}_3|_q = \mathscr{L}_{\mathrm{R}}(E_3)|_q - L_3|_q$, we get that

$$\hat{D}|_q \subset T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in (Q \backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M}).$$

Moreover, if $q = (x, (\hat{r}, \hat{y}); A) \in (Q \backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M})$ and if $\hat{\gamma} : [0, 1] \to \hat{N}$ is any curve with $\hat{\gamma}(0) = \hat{y}$, then one shows with exactly the same argument as in the proof of Proposition 5.31 that

$$\hat{\Gamma}(\hat{\gamma}, q)(t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \cap \pi_Q^{-1}(M_0 \times \hat{M}), \quad \forall t \in [0, 1].$$

In particular,

$$\exists q = (x, (\hat{r}, \hat{y}); A) \in (Q \backslash S_1) \cap \pi_Q^{-1}(M_0 \times \hat{M}) \implies \{x\} \times (\{\hat{r}\} \times \hat{N}) \subset \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)).$$

A similar argument shows that

$$D|_q \subset T|_q \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q), \quad \forall q \in (Q \backslash S_1) \cap \pi_Q^{-1}(M \times \hat{M}_0),$$

and that for all $q = ((r, y), \hat{x}; A) \in (Q \backslash S_1) \cap \pi_Q^{-1}(M \times \hat{M}_0)$ and $\gamma : [0, 1] \to N$ with $\gamma(0) = y$,

$$\Gamma(\gamma, q)(t) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \cap \pi_Q^{-1}(M \times \hat{M}_0), \quad \forall t \in [0, 1].$$

In particular,

$$\exists q = ((r, y), \hat{x}; A) \in (Q \backslash S_1) \cap \pi_Q^{-1}(M \times \hat{M}_0) \implies (\{r\} \times N) \times \{\hat{x}\} \subset \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)).$$

Everything so far is similar to the proof of Proposition 5.31 and continues to be so, with few minor changes (notably, here $\dim D = \dim \hat{D} = 2$ instead of 3). Suppose that $(M_1 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$. Take $q_1 = (x_1, \hat{x}_1; A_1) \in \pi_Q^{-1}(M_1 \times \hat{M}_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, with $x_1 = (r_1, y_1)$. If $\sigma(y)$ is the unique sectional curvature of N at y, we have

$$K_2(r_1, y_1) = \frac{\sigma(y_1) - (f'(r_1))^2}{f(r_1)^2} = K.$$

We go from here case by case.

(I) If N does not have constant curvature, there exists $y_2 \in N$ with $\sigma(y_2) \neq \sigma(y_1)$ and hence

$$K_2(r_1, y_2) = \frac{\sigma(y_2) - (f'(r_1))^2}{f(r_1)^2} \neq K,$$

i.e., $(r_1, y_2) \in M_0$. Since $q_1 \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset Q \backslash S_1$, we have by the above that

$$((r_1, y_2), \hat{x}_1) \in (\{r_1\} \times N) \times \{\hat{x}_1\} \subset \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_1)) = \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)),$$

and since $((r_1, y_2), \hat{x}_1) \in M_0 \times \hat{M}_0$, we get that which implies that $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$.

(II) Suppose that (N, h) has constant curvature C i.e. $\sigma(y) = C$ for all $y \in N$. We write $K_2(r, y) = K_2(r)$ on M since its value only depends on $r \in I$ and notice that for all $r \in I$,

$$\frac{\mathrm{d}K_2}{\mathrm{d}r} = -2\frac{f'(r)}{f(r)}(K_2(r) - K).$$

But $K_2(r_1) = K$, so by the uniqueness of solutions of ODEs, we get $K_2(r) = K$ for all $r \in I$ and hence (M, g) has constant curvature K.

Of course, regarding case (II), it is clear that if (M, g) has constant curvature K, then (N, h) has a constant curvature. Hence we have proved that if (M, g) does not have a constant curvature and if $(M_1 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$, then also $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$. The argument being symmetric in (M, g), $(\hat{M}, \hat{g})$, we also have that if $(\hat{M}, \hat{g})$ does not have a constant curvature and if $(M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$, then also $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$. Notice that (M°, g) and $(\hat{M}^\circ, \hat{g})$ cannot both have constant curvature, since this violates the assumption that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is not an integral manifold of $\mathcal{D}_{\mathrm{R}}$ (see Corollary 4.16 and Remark 4.17). We can now finish the proof by considering, again, different cases.

a) Assume that $(\hat{M}^\circ,\hat{g})$ has constant curvature equal then to K. We have $\hat{M}_0\cap\hat{M}^\circ=\emptyset$. If $E_2=\frac{\partial}{\partial r}$, then Hence, $\widetilde{\mathsf{Rol}}_q(\star X)=0$ for all $q\in Q^\circ=\pi_Q^{-1}(M^\circ\times\hat{M}^\circ)$, $X\in E_2^\perp$ while $\widetilde{\mathsf{Rol}}_q(\star E_2)=(-K_2(x)+K)\star E_2$. At $q_1=(x_1,\hat{x}_1;A_1)\in Q^\circ$, take an open neighbourhood U of x_1 and an ortonormal basis E_1,E_2,E_3 with $E_2=\frac{\partial}{\partial r}$ and let D_1 be a distribution on $\pi_{Q,M}^{-1}(U)$ spanned by

$$\mathscr{L}_{\mathrm{R}}(E_1),\mathscr{L}_{\mathrm{R}}(E_2),\mathscr{L}_{\mathrm{R}}(E_3),\nu((\cdot)\star E_2),L_1,L_3,$$

where L_1,L_3 are as in Proposition 5.5. Obviously, one defines in this way a 6-dimensional smooth distribution D_1 on the whole Q° and the above from of $\widetilde{\mathsf{Rol}}_q$, $q\in Q^\circ$, along with Proposition 5.5, case (ii), reveal that it is involutive (recall that $\Gamma^1_{(2,3)}=0$ there). Clearly, $\mathcal{D}_{\mathrm{R}}\subset D_1$ on Q° and since $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset Q^\circ$, we have $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\subset\mathcal{O}_{D_1}(q_0)$ and hence $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\le 6$. Because (M°,g) does not have constant curvature (as noticed previously), we have $M_0\cap M^\circ\neq\emptyset$ and thus $O:=\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\cap\pi_{Q,M}^{-1}(M_0)$ is a non-empty open subset of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. For every $q=(x,\hat{x};A)\in O$, one has $\widetilde{\mathsf{Rol}}_q(\star E_2)=(-K_2(x)+K)\star E_2\neq 0$ and hence that $\nu(A\star E_2)|_q\in T|_q\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Therefore, Proposition 5.5, case (i), implies that $D_1|_O$ is tangent to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. This gives $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)\ge 6$ and hence $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=6$.

b) If (M°,g) has constant curvature, then the argument of case a) with the roles of (M,g), $(\hat{M},\hat{g})$ interchanged, shows that $\dim\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=6$.

Hence we have proven (ii). For the rest of the cases, we may assume that neither (M°,g) nor $(\hat{M}^\circ,\hat{g})$ has constant curvature i.e. $M^\circ\cap M_0\neq\emptyset$, $\hat{M}^\circ\cap\hat{M}_0\neq\emptyset$.

c) Suppose $(M_0\times\hat{M}_0)\cap\pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))\neq\emptyset$ and let $q_1=(x_1,\hat{x}_1;A_1)\in\pi_Q^{-1}(M_0\times\hat{M}_0)\cap\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. We already know that $T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ contains vectors

$$\mathscr{L}_{\mathrm{R}}(E_1)|_{q_1},\mathscr{L}_{\mathrm{R}}(E_2)|_{q_1},\mathscr{L}_{\mathrm{R}}(E_3)|_{q_1},\nu(A\star E_2)|_{q_1},\nu((\hat{\star}\hat{E}_2)A)|_{q_1},L_1|_{q_1},L_3|_{q_1},\hat{L}_1|_{q_1},\hat{L}_3|_{q_1},$$

where

$$\begin{aligned}\hat{L}_1|_{q_1}&=\mathscr{L}_{\mathrm{NS}}(\hat{E}_1)|_{q_1}+\hat{\Gamma}^1_{(1,2)}(\hat{x}_1)\nu((\hat{\star}\hat{E}_3)A_1)|_{q_1},\\ \hat{L}_3|_{q_1}&=\mathscr{L}_{\mathrm{NS}}(\hat{E}_3)|_{q_1}-\hat{\Gamma}^1_{(1,2)}(\hat{x}_1)\nu((\hat{\star}\hat{E}_1)A_1)|_{q_1}.\end{aligned}$$

If $q_1\in Q_0$. these vectors span an 8-dimensional subspace of $T|_{q_1}\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, Indeed, by considering $X_{A_1},Z_{A_1},\hat{X}_{A_1},\hat{Z}_{A_1}$ and angles $\phi,\theta,\hat{\theta}$ as before, one has $\sin(\phi(q_1))\neq 0$ and

$$\begin{aligned}\nu((\hat{\star}\hat{E}_2)A_1)|_{q_1}=&\nu(A_1\star(A_1^{\overline{T}}\hat{E}_2))|_{q_1}\\ =&\sin(\phi(q_1))\nu(A_1\star X_{A_1})|_{q_1}+\cos(\phi(q_1))\nu(A_1\star E_2)|_{q_1},\\ c_\theta L_1|_{q_1}+s_\theta L_3|_{q_1}=&\mathscr{L}_{\mathrm{NS}}(X_{A_1})|_{q_1}-\Gamma^1_{(1,2)}(x_1)\nu(A_1\star Z_{A_1})|_{q_1},\\ -s_\theta L_1|_{q_1}+c_\theta L_3|_{q_1}=&\mathscr{L}_{\mathrm{NS}}(Z_{A_1})|_{q_1}+\Gamma^1_{(1,2)}(x_1)\nu(A_1\star X_{A_1})|_{q_1},\\ c_{\hat{\theta}}\hat{L}_1|_{q_1}+s_{\hat{\theta}}\hat{L}_3|_{q_1}=&\mathscr{L}_{\mathrm{NS}}(\hat{X}_{A_1})|_{q_1}+\hat{\Gamma}^1_{(1,2)}(x_1)\nu(A_1\star Z_{A_1})|_{q_1}\\ =&c_\phi\mathscr{L}_{\mathrm{NS}}(A_1X_{A_1})|_{q_1}-s_\phi\mathscr{L}_{\mathrm{NS}}(A_1E_2)|_{q_1}+\hat{\Gamma}^1_{(1,2)}(x_1)\nu(A_1\star Z_{A_1})|_{q_1},\\ -s_{\hat{\theta}}\hat{L}_1|_{q_1}+c_{\hat{\theta}}\hat{L}_3|_{q_1}=&\mathscr{L}_{\mathrm{NS}}(A_1Z_{A_1})|_{q_1}-\hat{\Gamma}^1_{(1,2)}(x_1)\nu(A_1\star(A^{\overline{T}}\hat{X}_{A_1}))|_{q_1}\\ =&\mathscr{L}_{\mathrm{NS}}(A_1Z_{A_1})|_{q_1}-\hat{\Gamma}^1_{(1,2)}(x_1)\big(c_\phi\nu(A_1\star X_{A_1})|_{q_1}-s_\phi\nu(A_1\star E_2)|_{q_1}\big).\end{aligned}$$

On the other hand, if $q_1 \in Q_1$, then since Q_1 is transversal to $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ at q_1, we can replace q_1 by a nearby $q_1' \in \pi_Q^{-1}(M_0 \times \hat{M}_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \cap Q_0$ and the above holds at q_1'. Therefore $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \geq 8$ and since we have also shown that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 8$, we have the equality.

d) Since $M^\circ \cap M_0 \neq \emptyset$, there is a $q_1 = (x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $x_1 \in M_0$. If $\hat{x}_1 \in \hat{M}_0$, one has that $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ and hence case c) implies that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 8$. If $\hat{x}_1 \notin \hat{M}_0$, then $\hat{x}_1 \in \overline{\hat{M}_1}$. Therefore, we may find a sequence $q_n' = (x_n', \hat{x}_n'; A_n') \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ such that $q_n' \to q_1$ and $\hat{x}_n' \in \hat{M}_1$. So for n large enough, we have $(x_n', \hat{x}_n') \in (M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0))$. Thus $(\hat{M}, \hat{g})$ does not have constant curvature and $(M_0 \times \hat{M}_1) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ which we have shown to imply that $(M_0 \times \hat{M}_0) \cap \pi_Q(\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)) \neq \emptyset$ from which the above case c) implies that $\dim \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \leq 8$.

The cases c) and d) above give (iii) and therefore the proof is complete. □

Remark 5.37 It is not difficult to see that Proposition 5.33 generalizes to higher dimension as follows. Keeping the same notations as before, let $(M, g) = (I, s_1) \times_f (N, h)$ and $(\hat{M}, \hat{g}) = (\hat{I}, s_1) \times_{\hat{f}} (\hat{N}, \hat{h})$, $I, \hat{I} \subset \mathbb{R}$, be warped products where (N, h) and $(\hat{N}, \hat{h})$ are now connected, oriented $(n-1)$-dimensional Riemannian manifolds. As before, let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ be such that if we write $x_0 = (r_0, y_0)$, $\hat{x}_0 = (\hat{r}_0, \hat{y}_0)$, then (50) and (51) hold true. Then, the exact argument of Proposition 5.33 yields that the orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ has dimension at most equal to $n(n+1)/2$. One can even have equality, if the $(n-1)$-dimensional manifolds (N, h) and $(\hat{N}, \hat{h})$ are such that that the corresponding $\widetilde{\mathsf{Rol}}_{q_0'}$ operator (in $(n-1)$-dimensional setting) is invertible at $q_0' = (y_0, \hat{y}_0; A_0') \in Q(N, \hat{N})$, where $A_0' : \frac{\partial}{\partial r}\big|_{x_0}^{\perp} \to \frac{\partial}{\partial r}\big|_{\hat{x}_0}^{\perp}$ is the restriction of A_0 and if we also assume that $f(r_0) = 1$, $\hat{f}(r_0) = 1$, an assumption that can always be satisfied after rescaling the metrics of (N, h) and $(\hat{N}, \hat{h})$.

A Fiber Coordinates and Control Theoretic Points of View

In this section we describe equations of the control system $(\Sigma)_R$ in terms of moving orthonormal frames. Assume that Let $F = (F_1, \ldots, F_n)$, $\hat{F} = (\hat{F}_1, \ldots, \hat{F}_n)$ be oriented orthonormal local frames of M and $\hat{M}$ defined on U and $\hat{U}$ respectively. We assume moreover that $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [0, 1]$, is an a.c. curve in Q sucht that $\gamma([0, 1]) \subset U$ and $\hat{\gamma}([0, 1]) \subset \hat{U}$.

Define for every $x \in U$ and $\hat{x} \in \hat{U}$ the linear maps

$$\Gamma : T|_x M \to \mathfrak{so}(n), \quad \Gamma(X)_i^j = g(\nabla_X F_i, F_j),$$
$$\hat{\Gamma} : T|_{\hat{x}} \hat{M} \to \mathfrak{so}(n), \quad \hat{\Gamma}(\hat{X})_i^j = \hat{g}(\hat{\nabla}_{\hat{X}} \hat{F}_i, \hat{F}_j).$$

Let $\mathcal{A} : [0, 1] \to \mathrm{SO}(n)$ be given by $\mathcal{A}(t) = \mathcal{M}_{F, \hat{F}}(A(t)) = [A_j^i(t)]$ i.e.,

$$(A(t)F_1|_{\gamma(t)}, \ldots, A(t)F_n|_{\gamma(t)}) = (\hat{F}_1|_{\hat{\gamma}(t)}, \ldots, \hat{F}_n|_{\hat{\gamma}(t)})\mathcal{A}(t).$$

Taking $\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}$ of this gives

$$
\begin{aligned}
&(\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(A(\cdot))F_1|_{\gamma(t)},\dots,\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(A(\cdot))F_n|_{\gamma(t)})+(A(t)\nabla_{\dot{\gamma}(t)}F_1,\dots,A(t)\nabla_{\dot{\gamma}(t)}F_n)\\
=&(\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{F}_1,\dots,\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{F}_n)\mathcal{A}(t)+(\hat{F}_1|_{\hat{\gamma}(t)},\dots,\hat{F}_n|_{\hat{\gamma}(t)})\dot{\mathcal{A}}(t),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&(\hat{F}_1|_{\hat{\gamma}(t)},\dots,\hat{F}_n|_{\hat{\gamma}(t)})\big(-\mathcal{A}(t)\Gamma(\dot{\gamma}(t))+\hat{\Gamma}(\dot{\hat{\gamma}}(t))\mathcal{A}(t)+\dot{\mathcal{A}}(t)\big)\\
=&(\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(A(\cdot))F_1|_{\gamma(t)},\dots,\overline{\nabla}_{(\dot{\gamma}(t),\dot{\hat{\gamma}}(t))}(A(\cdot))F_n|_{\gamma(t)}).
\end{aligned}
$$

Hence one sees that

$$
q(t)=(\gamma(t),\hat{\gamma}(t);A(t))\ \text{ satisfies Eq. (9)}\iff \dot{\mathcal{A}}(t)=\mathcal{A}(t)\Gamma(\dot{\gamma}(t))-\hat{\Gamma}(\dot{\hat{\gamma}}(t))\mathcal{A}(t).
$$

We newt show how to interpret $(\Sigma)_R$ as an affine driftless control system in $\pi_Q^{-1}(U\times\hat{U})$. Fix $q_0=(x_0,\hat{x}_0;A_0)\in\pi_Q^{-1}(U\times\hat{U})$. Note that there is an open subset $\mathcal{U}\subset L^1([0,1],\mathbb{R}^n)$ and a one-to-one correspondence between a.c. curves $\gamma:[0,1]\to U$ with $\gamma(0)=x_0$ and $\mathcal{U}$ given by

$$
\dot{\gamma}(t)=(F_1|_{\gamma(t)},\dots,F_n|_{\gamma(t)})\begin{pmatrix}u^1(t)\\ \vdots\\ u^n(t)\end{pmatrix},\quad (u^1,\dots,u^n)\in\mathcal{U}. \tag{53}
$$

The no-slip condition, Eq. (11) now becomes

$$
\dot{\hat{\gamma}}(t)=(\hat{F}_1|_{\hat{\gamma}(t)},\dots,\hat{F}_n|_{\hat{\gamma}(t)})\mathcal{A}(t)\begin{pmatrix}u^1(t)\\ \vdots\\ u^n(t)\end{pmatrix}, \tag{54}
$$

and, by the above, the no-spin condition, Eq. (9), becomes

$$
\dot{\mathcal{A}}(t)=\sum_{i=1}^n u^i(t)\Big(\mathcal{A}(t)\Gamma(F_i|_{\gamma(t)})-\sum_{j=1}^n A_i^j(t)\hat{\Gamma}(\hat{F}_j|_{\hat{\gamma}(t)})\mathcal{A}(t)\Big). \tag{55}
$$

Hence, the problem $(\Sigma)_R$ is equivalent on $\pi_Q^{-1}(U\times\hat{U})$ to the control system defined by Eqs. (53), (54), (55) where the controls $(u^1,\dots,u^n)$ belong to $\mathcal{U}\subset L^1([0,1],\mathbb{R}^n)$ and $\mathcal{A}(t)=\mathcal{M}_{F,\hat{F}}(A(t))=[A_j^i(t)]$. If $v=(v_1,\dots,v_n)\in\mathbb{R}^n$ we write $\langle F,v\rangle=\sum_{i=1}^n v_iF_i$ and $\Big\langle\hat{F},v\Big\rangle=\sum_{i=1}^n v_i\hat{F}_i$. With this notation, if we write $u=(u^1,\dots,u^n)$, we write the system (53), (54), (55) more compactly as

$$
\begin{cases}
\dot{\gamma}(t)=\left\langle F|_{\gamma(t)},u(t)\right\rangle,\\
\dot{\hat{\gamma}}(t)=\Big\langle \hat{F}|_{\hat{\gamma}(t)}\mathcal{A}(t),u(t)\Big\rangle,\\
\dot{\mathcal{A}}(t)=\mathcal{A}(t)\Gamma\big(\left\langle F|_{\gamma(t)},u(t)\right\rangle\big)-\hat{\Gamma}\big(\Big\langle \hat{F}|_{\hat{\gamma}(t)}\mathcal{A}(t),u(t)\Big\rangle\big)\mathcal{A}(t).
\end{cases}
$$

B The Rolling Problem Embedded in $\mathbb{R}^N$

In this section, we compare the rolling model defined by the state space $Q = Q(M,\hat{M})$, whose dynamics is governed by the conditions (10)-(11) (or, equivalently, by $\mathcal{D}_{\mathrm{R}}$), with the rolling model of two n-dimensional manifolds embedded in $\mathbb{R}^N$ as given in [32] (Appendix B). See also [21], [13].

Let us first fix $N \in \mathbb{N}$ and introduce some notations. The special Euclidean group of $\mathbb{R}^N$ is the set $\mathrm{SE}(N) := \mathbb{R}^N \times \mathrm{SO}(N)$ equipped with the group operation $\star$ given by

$$(p,A)\star(q,B) = (Aq+p, AB), \quad (p,A),(q,B)\in \mathrm{SE}(N).$$

We identify $\mathrm{SO}(N)$ with the subgroup $\{0\}\times\mathrm{SO}(N)$ of $\mathrm{SE}(N)$, while $\mathbb{R}^N$ is identified with the normal subgroup $\mathbb{R}^N \times \{\mathrm{id}_{\mathbb{R}^N}\}$ of $\mathrm{SE}(N)$. With these identifications, the action $\star$ of the subgroup $\mathrm{SO}(N)$ on the normal subgroup $\mathbb{R}^N$ is given by

$$(p,A)\star q = Aq+p, \quad (p,A)\in \mathrm{SE}(N),\ p\in\mathbb{R}^N.$$

Let $\mathcal{M}$ and $\hat{\mathcal{M}} \subset \mathbb{R}^N$ be two (embedded) submanifolds of dimension n. For every $z\in\mathcal{M}$, we identify $T|_z\mathcal{M}$ with a subspace of $\mathbb{R}^N$ (the same holding in the case of $\hat{\mathcal{M}}$) i.e., elements of $T|_z\mathcal{M}$ are derivatives $\dot{\sigma}(0)$ of curves $\sigma : I \to M$ with $\sigma(0)=z$ ($I \ni 0$ a nontrivial real interval).

The *rolling of $\mathcal{M}$ against $\hat{\mathcal{M}}$ without slipping or twisting* in the sense of [32] is realized by a smooth curves $G : I \to \mathrm{SE}(N)$; $G(t) = (p(t),U(t))$ (I a nontrivial real interval) called the *rolling map* and $\sigma : I \to \mathcal{M}$ called the *development* curve such that the following conditions (1)-(3) hold for every $t\in I$:

(1) (a) $\hat{\sigma}(t) := G(t)\star\sigma(t) \in \hat{\mathcal{M}}$ and
(b) $T|_{\hat{\sigma}(t)}(G(t)\star\mathcal{M}) = T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}$.

(2) No-slip: $\dot{G}(t)\star\sigma(t) = 0$.

(3) No-twist: (a) $\dot{U}(t)U(t)^{-1}T|_{\hat{\sigma}(t)}\hat{\mathcal{M}} \subset (T|_{\hat{\sigma}(t)}\hat{\mathcal{M}})^\perp$ (tangential no-twist),
(b) $\dot{U}(t)U(t)^{-1}(T|_{\hat{\sigma}(t)}\hat{\mathcal{M}})^\perp \subset T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}$ (normal no-twist).

The orthogonal complements are taken w.r.t. the Euclidean inner product of $\mathbb{R}^N$. In condition (2) we define the action '$\star$' of $\dot{G}(t) = (\dot{U}(t),\dot{p}(t))$ on $\mathbb{R}^N$ by the same formula as for the action '$\star$' of $\mathrm{SE}(N)$ on $\mathbb{R}^N$.

We next consider two classical cases of rolling and interpret the no-twist conditions in these cases.

Example B.1 (i) Suppose $N=3$, $n=2$ i.e., $\mathcal{M},\hat{\mathcal{M}}$ are surfaces of $\mathbb{R}^3$. Assuming that they are oriented, there exist smooth normal vector fields N, $\hat{N}$ of $\mathcal{M}$ and $\hat{\mathcal{M}}$ respectively. For a given t, choose oriented orthonormal frame $\hat{X},\hat{Y}\in T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}$ and $\hat{a},\hat{b},\hat{c}\in\mathbb{R}$ such that $\dot{U}(t)U(t)^{-1}\in\mathfrak{so}(3)$ can be written as

$$\dot{U}(t)U(t)^{-1} = \hat{a}(\hat{N}|_{\hat{\sigma}(t)}\times) + \hat{b}(\hat{X}\times) + \hat{c}(\hat{Y}\times),$$

where $\times$ denotes the cross product in $\mathbb{R}^3$ and for a vector $V\in\mathbb{R}^3$ we denote by $(V\times)$ the element of $\mathfrak{so}(3)$ given by $W\in\mathbb{R}^3 \mapsto V\times W\in\mathbb{R}^3$. It is now easy to see, by applying $\dot{U}(t)U(t)^{-1}$ to $\hat{X},\hat{Y}$, that the tangential no-twist condition (3)-(a) is equivalent to the fact that $\hat{a}=0$ i.e.,

$$\dot{U}(t)U(t)^{-1} \quad \text{does not contain } (\hat{N}|_{\hat{\sigma}(t)}\times)\text{-component.}$$

This is what is intuitively understood by "no spinning" since it is the $(\hat{N}|_{\hat{\sigma}(t)}\times)$ component $\hat{a}$ of $\dot{U}(t)U(t)^{-1}$ that mesures the instantaneous speed of rotation of $\mathcal{M}$ about the axis $\hat{N}|_{\hat{\sigma}(t)}$ at the corresponding point of contact. Notice also that

$$\dot{U}(t)U(t)^{-1}\hat{N}|_{\hat{\sigma}(t)} = -\hat{b}\hat{Y} + \hat{c}\hat{X} \in T|_{\hat{\sigma}(t)}\hat{\mathcal{M}},$$

so the normal no-twist condition (3)-(b) is automatically satisfied. This example can be easily generalized to any case of oriented hypersurfaces i.e. when $N=n+1$.

(ii) Suppose now that $N=3$ and $n=1$ i.e. $\mathcal{M},\hat{\mathcal{M}}$ are regular curves in $\mathbb{R}^3$. Without loss of generality, we assume that $\|\dot{\sigma}(t)\| = 1$, hence also $\left\|\dot{\hat{\sigma}}(t)\right\| = 1$. Let $\hat{X},\hat{Y} \in \dot{\hat{\sigma}}(t)^{\perp}$ such that $\hat{X},\hat{Y},\dot{\hat{\sigma}}(t)$ is an oriented orthonormal frame in $\mathbb{R}^3$. One may write $\dot{U}(t)U(t)^{-1} \in \mathfrak{so}(3)$ as

$$\dot{U}(t)U(t)^{-1} = \hat{a}(\dot{\hat{\sigma}}(t)\times) + \hat{b}(\hat{X}\times) + \hat{c}(\hat{Y}\times).$$

Since then

$$\dot{U}(t)U(t)^{-1}\dot{\hat{\sigma}} = -\hat{b}\hat{Y} + \hat{c}\hat{X},$$

the tangential no-twist condition (3)-(a) is trivially satisfied. As for the normal no-twist condition (3)-(b), one sees that it is equivalent to $\hat{a}=0$ i.e.,

$$\dot{U}(t)U(t)^{-1} \quad \text{does not contain } (\dot{\hat{\sigma}}(t)\times)\text{-component.}$$

Intuitively this means that the instantaneous speed of rotation $\hat{a}$ of $\mathcal{M}$ about the axis $\dot{\hat{\sigma}}(t)$ is zero at the point of contact, so $\mathcal{M}$ does not turn around $\hat{\mathcal{M}}$.

The two manifolds M and $\hat{M}$ are embedded inside $\mathbb{R}^N$ by embeddings $\iota : M \to \mathbb{R}^N$ and $\hat{\iota} : \hat{M} \to \mathbb{R}^N$ and their metrics g and $\hat{g}$ are induced from the Euclidean metric s_N of $\mathbb{R}^N$ i.e., $g = \iota^* s_N$ and $\hat{g} = \hat{\iota}^* s_N$. In the above setting, we take now $\mathcal{M} = \iota(M)$, $\hat{\mathcal{M}} = \hat{\iota}(\hat{M})$. For $z \in \mathcal{M}$ and $\hat{z} \in \hat{\mathcal{M}}$, consider the linear orthogonal projections

$$P^T : T|_z\mathbb{R}^N \to T|_z\mathcal{M} \text{ and } P^{\perp} : T|_z\mathbb{R}^N \to T|_z\mathcal{M}^{\perp},$$

and

$$\hat{P}^T : T|_{\hat{z}}\mathbb{R}^N \to T|_{\hat{z}}\hat{\mathcal{M}} \text{ and } \hat{P}^{\perp} : T|_{\hat{z}}\mathbb{R}^N \to T|_{\hat{z}}\hat{\mathcal{M}}^{\perp},$$

respectively. For $X \in T|_z\mathbb{R}^N$ and $Y \in \Gamma(\pi_{T\mathbb{R}^N}|_{\mathcal{M}})$ (here $\pi_{T\mathbb{R}^N}|_{\mathcal{M}}$ is the pull-back bundle of $T\mathbb{R}^N$ over $\mathcal{M}$), we use $\nabla^{\perp}_X Y$ to denote $P^{\perp}(\nabla^{s_N}_X Y)$ and one writes similarly $\hat{\nabla}^{\perp}_{\hat{X}}\hat{Y} = \hat{P}^{\perp}(\nabla^{s_N}_{\hat{X}}\hat{Y})$ for $\hat{X} \in T|_{\hat{z}}\mathbb{R}^N$ and $Y \in \Gamma(\pi_{T\mathbb{R}^N}|_{\hat{\mathcal{M}}})$. We notice that, for any $z \in \mathcal{M}$, $X \in T|_z\mathcal{M}$ and $Y \in \mathrm{VF}(\mathcal{M})$, we have

$$\nabla^{s_N}_X Y = \iota_*(\nabla_{\iota_*^{-1}(X)}\iota_*^{-1}(Y)) + \nabla^{\perp}_X Y,$$

and similarly on $\hat{\mathcal{M}}$. Notice that $\nabla^{\perp}$ and $\hat{\nabla}^{\perp}$ determine (by restriction) connections of vector bundles $\pi_{T\mathcal{M}^{\perp}} : T\mathcal{M}^{\perp} \to \mathcal{M}$ and $\pi_{T\hat{\mathcal{M}}^{\perp}} : T\hat{\mathcal{M}}^{\perp} \to \hat{\mathcal{M}}$. These connections can then be used in an obvious way to determine a connection $\overline{\nabla}^{\perp}$ on the vector bundle

$$\pi_{(T\mathcal{M}^{\perp})^*\otimes T\mathcal{M}^{\perp}} : (T\mathcal{M}^{\perp})^* \otimes T\mathcal{M}^{\perp} \to \mathcal{M} \times \hat{\mathcal{M}}.$$

Let us take any rolling map $G : I \to \mathrm{SE}(N)$, $G(t) = (p(t), U(t))$ and development curve $\sigma : I \to \mathcal{M}$ and define $x = \iota^{-1} \circ \sigma$. We will go throught the meaning of each of the above conditions (1)-(3).

(1) (a) Since $\hat{\sigma}(t) \in \hat{\mathcal{M}}$, we may define a smooth curve $\hat{x} := \hat{\iota}^{-1} \circ \hat{\sigma}$ in $\hat{M}$.
(b) One easily sees that

$$U(t)T|_{\hat{\sigma}(t)}\mathcal{M} = T|_{\hat{\sigma}(t)}(G(t) \star \mathcal{M}) = T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}.$$

Thus $A(t) := \hat{\iota}_*^{-1} \circ U(t) \circ \iota_*|_{T|_{x(t)}M}$ defines a map $T|_{x(t)}M \to T|_{\hat{x}(t)}\hat{M}$, which is also orthogonal i.e., $A(t) \in Q|_{(x(t),\hat{x}(t))}$ for all t. Moreover, if $B(t) := U(t)|_{T|_{\sigma(t)}\mathcal{M}^{\perp}}$, then $B(t)$ is a map $T|_{\sigma(t)}\mathcal{M}^{\perp} \to T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}^{\perp}$ and, by a slight abuse of notation, we can write $U(t) = A(t) \oplus B(t)$. Thus Condition (1) just determines a smooth curve $t \mapsto (x(t), \hat{x}(t); A(t))$ inside the state space $Q = Q(M, \hat{M})$.

(2) We compute

$$\begin{aligned} 0 =& \dot{G}(t) \star \sigma(t) = \dot{U}(t)\sigma(t) + \dot{p}(t) \\ =& \frac{\mathrm{d}}{\mathrm{d}t}(G(t) \star \sigma(t)) - U(t)\dot{\sigma}(t) = \dot{\hat{\sigma}}(t) - U(t) \circ \iota_* \circ \iota_*^{-1} \circ \dot{\sigma}(t), \end{aligned}$$

which, once composed with $\hat{\iota}_*^{-1}$ from the left, gives $0 = \dot{\hat{x}}(t) - A(t)\dot{x}(t)$. This is exactly the no-slip condition, Eq. (11).

(3) Notice that, on $\mathbb{R}^N \times \mathbb{R}^N = \mathbb{R}^{2N}$, the sum metric $s_N \oplus s_N$ is just s_{2N}. Moreover, if $\gamma : I \to \mathbb{R}^N$ is a smooth curve, then smooth vector fields $X : I \to T(\mathbb{R}^N)$ along γ can be identified with smooth maps $X : I \to \mathbb{R}^N$ and with this observation one has: $\dot{X}(t) = \nabla^{s_N}_{\dot{\gamma}(t)} X$.
(a) Since $U(t) = A(t) \oplus B(t)$, we get, for $t \mapsto \hat{X}(t) \in T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}$, that

$$\begin{aligned} &\dot{U}(t)U(t)^{-1}\hat{X}(t) = \nabla^{s_{2N}}_{(\dot{\sigma},\dot{\hat{\sigma}})(t)}\hat{X}(\cdot) - U(t)\nabla^{s_{2N}}_{(\dot{\sigma},\dot{\hat{\sigma}})(t)}(U(\cdot)^{-1}\hat{X}(\cdot)) \\ =&P^T\big(\hat{\nabla}^{s_N}_{\dot{\hat{\sigma}}(t)}\hat{X}(\cdot)\big) + \hat{\nabla}^{\perp}_{\dot{\hat{\sigma}}(t)}\hat{X}(\cdot) \\ &- U(t)\big(P^T\big(\nabla^{s_N}_{\dot{\sigma}(t)}(A(\cdot)^{-1}\hat{X}(\cdot))\big) + \nabla^{\perp}_{\dot{\sigma}(t)}(A(\cdot)^{-1}\hat{X}(\cdot))\big) \\ =&\big(\overline{\nabla}_{(\dot{x},\dot{\hat{x}})(t)}A(\cdot)\big)A(t)^{-1}(\hat{\iota}_*^{-1}\hat{X}(t)) + \big(\hat{\nabla}^{\perp}_{\dot{\hat{\sigma}}(t)}\hat{X}(\cdot) - B(t)\nabla^{\perp}_{\dot{\sigma}(t)}(A(\cdot)^{-1}\hat{X}(\cdot))\big), \end{aligned}$$

from which it is clear that the tangential no-twist condition corresponds to the condition that $\overline{\nabla}_{(\dot{x}(t),\dot{\hat{x}}(t))}A(\cdot) = 0$. This means that $t \mapsto (x(t), \hat{x}(t); A(t))$ is tangent to $\mathcal{D}_{\mathrm{NS}}$ for all $t \in I$. Thus, the tangential no-twist condition (3)-(a) is equivalent to the no-spinning condition, Eq. (9).
(b) Choose $t \mapsto \hat{X}^{\perp}(t) \in T|_{\hat{\sigma}(t)}\hat{\mathcal{M}}^{\perp}$ and calculate as above

$$\begin{aligned} \dot{U}(t)U(t)^{-1}\hat{X}^{\perp}(t) = &P^T(\nabla^{s_N}_{\dot{\hat{\sigma}}(t)}\hat{X}^{\perp}(\cdot)) + \hat{\nabla}^{\perp}\dot{\hat{\sigma}}(t) \\ &- U(t)\big(P^T\big(\nabla^{s_N}_{\dot{\sigma}(t)}(B(\cdot)^{-1}\hat{X}(\cdot))\big) + \nabla^{\perp}_{\dot{\sigma}(t)}(B(\cdot)^{-1}\hat{X}(\cdot))\big) \\ =&\Big(P^T(\nabla^{s_N}_{\dot{\hat{\sigma}}(t)}\hat{X}^{\perp}(\cdot) - A(t)P^T\big(\nabla^{s_N}_{\dot{\sigma}(t)}(B(\cdot)^{-1}\hat{X}(\cdot))\big)\Big) \\ &+ \big(\overline{\nabla}^{\perp}_{(\dot{\sigma}(t),\dot{\hat{\sigma}}(t))}B(\cdot)\big)B(t)^{-1}\hat{X}(t), \end{aligned}$$

and hence we see that the normal no-twist condition (3)-(b) corresponds to the condition that

$$\overline{\nabla}^{\perp}_{(\sigma(t),\hat{\sigma}(t))}B(\cdot)=0,\quad \forall t.$$

In a similar spirit to how Definition 3.5 was given, one easily sees that this condition just amounts to say that B maps parallel translated normal vectors to $\mathcal{M}$ to parallel translated normal vectors to $\hat{\mathcal{M}}$. More precisely, if $X_0\in T\mathcal{M}^{\perp}$ and $X(t)=(P^{\nabla^{\perp}})^t_0(\sigma)X_0$ is a parallel translate of X_0 along σ w.r.t. to the connection $\nabla^{\perp}$ (notice that $X(t)\in T|_{\sigma(t)}\mathcal{M}^{\perp}$ for all t), then the normal no-twist condition (3)-(b) requires that $t\mapsto B(t)X(t)$ (which is the same as $U(t)X(t)$) is parallel to $t\mapsto\hat{\sigma}(t)$ w.r.t the connection $\hat{\nabla}^{\perp}$ i.e., for all t,

$$B(t)((P^{\nabla^{\perp}})^t_0(\sigma)X_0)=(P^{\hat{\nabla}^{\perp}})^t_0(\hat{\sigma})(B(0)X_0).$$

We formulate the preceding remarks to a proposition.

Proposition B.2 Let $\iota:M\to\mathbb{R}^N$ and $\hat{\iota}:\hat{M}\to\mathbb{R}^N$ be smooth embeddings and let $g=\iota^*(s_N)$ and $\hat{g}=\hat{\iota}^*(s_N)$. Fix points $x_0\in M$, $\hat{x}_0\in\hat{M}$ and an element $B_0\in\mathrm{SO}(T|_{\iota(x_0)}\mathcal{M}^{\perp},T|_{\hat{\iota}(\hat{x}_0)}\hat{\mathcal{M}}^{\perp})$. Then, there is a bijective correspondence between the smooth curves $t\mapsto(x(t),\hat{x}(t);A(t))$ of Q tangent to $\mathcal{D}_{\mathrm{NS}}$ (resp. $\mathcal{D}_{\mathrm{R}}$), satisfying $(x(0),\hat{x}(0))=(x_0,\hat{x}_0)$ and the pairs of smooth curves $t\mapsto G(t)=(p(t),U(t))$ of $\mathrm{SE}(N)$ and $t\mapsto\sigma(t)$ of $\mathcal{M}$ which satisfy the conditions (1), (3) (resp. (1),(2),(3) i.e., rolling maps) and $U(0)|_{T|_{\sigma(0)}\mathcal{M}^{\perp}}=B_0$.

Proof. Let $t\mapsto q(t)=(x(t),\hat{x}(t);A(t))$ to be a smooth curve in Q such that $(x(0),\hat{x}(0))=(x_0,\hat{x}_0)$. Denote $\sigma=\iota\circ x$, $\hat{\sigma}=\hat{\iota}\circ\hat{x}$ and let $B(t)=(P^{\overline{\nabla}^{\perp}})^t_0((\sigma,\hat{\sigma}))B_0$ be the parallel translate of B_0 along $t\mapsto(\sigma(t),\hat{\sigma}(t))$ w.r.t the connection $\overline{\nabla}^{\perp}$. We define

$$U(t):=(\hat{\iota}_*\circ A(t)\circ\iota_*^{-1})\oplus B(t):T|_{\sigma(t)}\mathcal{M}\to T|_{\hat{\sigma}(t)}\hat{\mathcal{M}},$$

and $p(t)=\hat{\sigma}(t)-U(t)\sigma(t)$. Then, by the above remarks, the smooth curve $t\mapsto G(t)=(p(t),U(t))$ satisfies Conditions (1),(3) (resp. (1),(2),(3)) if $t\mapsto q(t)$ is tangent to $\mathcal{D}_{\mathrm{NS}}$ (resp. $\mathcal{D}_{\mathrm{R}}$). This clearly gives the claimed bijective correspondence. □

C Special Manifolds in 3D Riemanniann Geometry

C.1 Preliminaries

On an oriented Riemannian manifold (M,g), the Hodge-dual $\star_M$ is defined as the linear map uniquely given by

$$\star_M:\wedge^kT|_xM\to\wedge^{n-k}T|_xM;\quad \star_M(X_1\wedge\cdots\wedge X_k)=X_{k+1}\wedge\cdots\wedge X_n,$$

with $x\in M$, $k=0,\dots,n=\dim M$ and $X_1,\dots,X_n\in T|_xM$ any oriented basis. For an oriented Riemannian manifold (M,g) and $x\in M$, let $\mathfrak{so}(T|_xM)$ be the set of g-antisymmetric linear maps $T|_xM\to T|_xM$. and writes $\mathfrak{so}(M)$ as the disjoint union of $\mathfrak{so}(T|_xM)$, $x\in M$. If $A,B\in\mathfrak{so}(T|_xM)$, we define

$$[A,B]_{\mathfrak{so}}:=A\circ B-B\circ A\in\mathfrak{so}(T|_xM).$$

Also, we define the following natural isomorphism ϕ by

$$\phi : \wedge^2 TM \to \mathfrak{so}(M); \quad \phi(X \wedge Y) := g(\cdot, X)Y - g(\cdot, Y)X.$$

Using this isomorphism, the curvature tensor R of (M, g) at $x \in M$, is the linear map given by

$$\mathcal{R} : \wedge^2 T|_x M \to \wedge^2 T|_x M; \quad \mathcal{R}(X \wedge Y) := \phi^{-1}(R(X, Y)),$$

where $X, Y \in T|_x M$. Here of course $R(X, Y)$, as an element of $T^*|_x M \otimes T|_x M$, belongs to $\mathfrak{so}(T|_x M)$. It is a standard fact that $\mathcal{R}$ is a symmetric map when $\wedge^2 T|_x M$ is endowed with the inner product, also written as g,

$$g(X \wedge Y, Z \wedge W) := g(X, Z)g(Y, W) - g(X, W)g(Y, Z).$$

For $A, B \in \mathfrak{so}(T|_x M)$, $\mathrm{tr}(AB) = g(\phi^{-1}(A), \phi^{-1}(B))$. The map $\mathcal{R}$ is the curvature operator and we will, with a slight abuse of notation, write it as R.

If $\dim M = 3$, then $\star_M^2 = \mathrm{id}$ when $\star_M$ is the map $\wedge^2 TM \to TM$ and $TM \to \wedge^2 TM$. Let $X, Y, Z \in T|_x M$ be an orthonormal positively oriented basis. Then

$$\star_M(X \wedge Y) = Z, \quad \star_M(Y \wedge Z) = X, \quad \star_M(Z \wedge X) = Y.$$

In terms of this basis X, Y, Z one has

$$\star_M \phi^{-1} \begin{pmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\gamma \\ -\beta & \gamma & 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ \beta \\ \alpha \end{pmatrix}.$$

Lemma C.1 If (M, g) is a 3-dimensional oriented Riemannian manifold and $x \in M$.

(i) Then each 2-vector $\xi \in \wedge^2 T|_x M$ is pure i.e. there exist $X, Y \in T|_x M$ such that $\xi = X \wedge Y$.

(ii) For every $X, Y \in T|_x M$ one has

$$[\phi(\star_M X), \phi(\star_M Y)]_{\mathfrak{so}} = \phi(X \wedge Y).$$

C.2 Manifolds of class M_β

In this subsection, we define and investigate some properties of special type of 3-dimensional manifolds. Following the paper [2] we make the following definition.

Definition C.2 A 3-dimensional manifold M is called a *contact manifold of type* $(\kappa, 0)$ where $\kappa \in C^\infty(M)$ if there are everywhere linearly independent vector fields $F_1, F_2, F_3 \in \mathrm{VF}(M)$ and smooth functions $c, \gamma_1, \gamma_3 \in C^\infty(M)$ such that

$$\begin{aligned}
[F_1, F_2] =& cF_3, \\
[F_2, F_3] =& cF_1, \\
[F_3, F_1] =& -\gamma_1 F_1 + F_2 - \gamma_3 F_3,
\end{aligned}$$

and

$$-\kappa = F_3(\gamma_1) - F_1(\gamma_3) + (\gamma_1)^2 + (\gamma_3)^2 - c.$$

The frame F_1, F_2, F_3 is said to be an *(normalized) adapted frame of* M and c, γ_1, γ_2 the corresponding structure functions.

Remark C.3 Contact manifolds in 3D are essentially classified by two functions κ, χ defined on these manifolds. Thus one could say in general that a contact manifold is of class (κ, χ). We are interested here only in the case where $\chi = 0$. For information on the classification of contact manifolds, definition of χ and references, see [2].

One may define on such a manifold a Riemannian metric in a natural way by declaring F_1, F_2, F_3 orthogonal. The structure of the connection table (see section C.4) and the eigenvalues of the corresponding curvature tensor are given in the following lemma, which is a direct consequence of Koszul's formula.

Lemma C.4 Let M be a contact manifold of type $(\kappa, 0)$ with adapted frame F_1, F_2, F_3 and structure functions c, γ_1, γ_2. If g is the unique Riemannian metric which makes F_1, F_2, F_3 orthonormal, then the connection table w.r.t. F_1, F_2, F_3 is

$$\Gamma = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \gamma_1 & c - \frac{1}{2} & \gamma_3 \\ 0 & 0 & \frac{1}{2} \end{pmatrix},$$

Moreover, at each point, $\star F_1, \star F_2, \star F_3$ (with $\star$ the Hodge dual) are eigenvectors of the curvature tensor R with eigenvalues $-K, -K_2(\cdot), -K$, respectively, where

$$\begin{aligned} K =& \frac{1}{4}, \quad \text{(constant)}, \\ K_2(x) =& \kappa(x) - \frac{3}{4}, \quad x \in M. \end{aligned}$$

To justify somewhat our next definition, we make the following remark.

Remark C.5 Notice that if $\beta \in \mathbb{R}$, $\beta \neq 0$ and $g_\beta := \beta^{-2} g$ then the Koszul-formula gives,

$$2g_\beta(\nabla^{g_\beta}_{F_i} F_j, F_k) = \beta^{-2} g([F_i, F_j], F_k) - \beta^{-2} g([F_i, F_k], F_j) - \beta^{-2} g([F_j, F_k], F_i) = 2\beta^{-2} \Gamma^i_{(j,k)},$$

because $g_\beta(F_i, F_j) = \beta^{-2}\delta_{ij}$. Then, $E_i := \beta F_i$, $i = 1, 2, 3$, is a g_β-orthonormal basis and if $(\Gamma_\beta)^i_{(j,k)} = g_\beta(\nabla_{E_i} E_j, E_k)$, then for every i, j, k.

$$\beta^{-3} (\Gamma_\beta)^i_{(j,k)} = \beta^{-3} g_\beta(\nabla^{g_\beta}_{E_i} E_j, E_k) = g_\beta(\nabla^{g_\beta}_{F_i} F_j, F_k) = \beta^{-2} \Gamma^i_{(j,k)},$$

i.e. $(\Gamma_\beta)^i_{(j,k)} = \beta \Gamma^i_{(j,k)}$.

Definition C.6 A 3-dimensional Riemannian manifold (M, g) is said to belong to *class* $\mathcal{M}_\beta$, for $\beta \in \mathbb{R}$, if there exists an orthonormal frame $E_1, E_2, E_3 \in \mathrm{VF}(M)$ w. r. t. which the connection table is

$$\Gamma = \begin{pmatrix} \beta & 0 & 0 \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ 0 & 0 & \beta \end{pmatrix}.$$

In this case the frame E_1, E_2, E_3 is called an *adapted frame of* (M, g).

Remark C.7 For a given $\beta \in \mathbb{R}$, one can say that a Riemannian space (M, g) is *locally of class* $\mathcal{M}_\beta$, if every $x \in M$ has an open neighbourhood U such that $(U, g|_U)$ is of class $\mathcal{M}_\beta$. Since we are interested in local results, we usually speak of manifolds of (globally) class $\mathcal{M}_\beta$.

Lemma C.8 If $\beta \neq 0$ and (M,g) is of class $\mathcal{M}_\beta$ with an adapted frame, then $\star E_1, \star E_2, \star E_3$ are eigenvectors of R with eigenvalues $-\beta^2, -K_2(\cdot), -\beta^2$, where

$$-K_2(x) = \beta^2 + E_3(\Gamma^1_{(3,1)}) - E_1(\Gamma^3_{(3,1)}) + (\Gamma^1_{(3,1)})^2 + (\Gamma^3_{(3,1)})^2 - 2\beta\Gamma^2_{(3,1)}, \quad x \in M.$$

Proof. Immediate from Proposition C.17, Eq. (58). □

Next lemma is the converse of what has been done before the above definition.

Lemma C.9 Let (M,g) be of class $\mathcal{M}_\beta$, $\beta \neq 0$, with an adapted frame E_1, E_2, E_3. Then M is a contact manifold of type $(\kappa, 0)$ with (normalized) adapted frame $F_i := \frac{1}{2\beta}E_i$, $i = 1,2,3$. Moreover, for $x \in M$, κ and the structure functions c, γ_1, γ_3 are given by

$$c = \frac{\beta + \Gamma^2_{(3,1)}}{2\beta}, \quad \gamma_1 = \frac{\Gamma^1_{(3,1)}}{2\beta}, \quad \gamma_3 = -\frac{\Gamma^3_{(3,1)}}{2\beta}, \kappa(x) = \frac{K_2(x)}{4\beta^2} + \frac{3}{4}.$$

Proof. From the torsion freeness of the Levi-Civita connection on (M,g) and from the connection table w.r.t. E_1, E_2, E_3, we get

$$\begin{aligned}
[E_1, E_2] &= (\beta + \Gamma^2_{(3,1)})E_3, \\
[E_2, E_3] &= (\beta + \Gamma^2_{(3,1)})E_1, \\
[E_3, E_1] &= -\Gamma^1_{(3,1)}E_1 + 2\beta E_2 - \Gamma^3_{(3,1)}E_3.
\end{aligned}$$

From this and the fact that $\beta \neq 0$, the claims are immediate. □

Remark C.10 (i) Note that the classes $\mathcal{M}_\beta$ and $\mathcal{M}_{-\beta}$ are the same. Indeed, if (M,g) is of class $\mathcal{M}_\beta$ and E_1, E_2, E_3 is an adapted orthonormal frame, then (M,g) is of class $\mathcal{M}_{-\beta}$ with a adapted frame F_1, F_2, F_3 where $F_1 = E_3$, $F_3 = E_1$ (i.e., the change of orientation of E_1, E_3 plane moves from $\mathcal{M}_\beta$ to $\mathcal{M}_{-\beta}$). It would then be better to speak of Riemannian manifolds of class $\mathcal{M}_\beta$ with $\beta \geq 0$ or of class $\mathcal{M}_{|\beta|}$.

(ii) If one has a Riemannian manifold (M,g) of class $\mathcal{M}_\beta$, then scaling the metric by $\lambda \neq 0$ one gets a Riemannian manifold $(M, \lambda^2 g)$ of class $\mathcal{M}_{\beta/\lambda}$. This follows from Remark C.5 above.

Remark C.11 If (M,g) is of class $\mathcal{M}_0$, then since $\beta = 0$ and $\Gamma^1_{(1,2)} = 0$, one deduces e.g. from Theorem C.14 that (M,g) is locally a warped product. Conversely, a Riemannian product manifold is locally of class $\mathcal{M}_0$. Hence there are many non-isometric spaces of class $\mathcal{M}_0$.

To conclude this subsection, we will show that that for every $\beta \in \mathbb{R}$ there exist 3-dimensional Riemannian manifolds of class $\mathcal{M}_\beta$ which are not isometric. See also [2].

Example C.12 (i) Let M be $\mathrm{SO}(3)$. There exists left-invariant vector fields E_1, E_2, E_3 such that

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

Hence with the metric g rendering E_1, E_2, E_3 orthonormal, we get a space (M,g) of class $\mathcal{M}_{1/2}$. By the definition of κ and Lemma C.9 we have $\kappa = 1$ and $K_2 = \frac{1}{4}$.

(ii) Let M be the Heisenberg group H_3. There exists left-invariant vector fields E_1, E_2, E_3 such that

$$[E_1, E_2] = 0, \quad [E_2, E_3] = 0, \quad [E_3, E_1] = E_2.$$

Then, M endowed with the metric for which E_1, E_2, E_3 are orthonormal, is of class $\mathcal{M}_{1/2}$ and $\kappa = 0$, $K_2 = -\frac{3}{4}$.

(iii) Let M be $\mathrm{SL}(2)$. There exists left-invariant vector fields such that

$$[E_1, E_2] = -E_3, \quad [E_2, E_3] = -E_1, \quad [E_3, E_1] = E_2.$$

If g is a metric with respect to which E_1, E_2, E_3 are orthonormal, then M is of class $\mathcal{M}_{1/2}$, with $\kappa = -1$ and $K_2 = -\frac{7}{4}$.

Note that if one takes the "usual" basis of $\mathfrak{sl}(2)$ as a, b, c satisfying,

$$[c, a] = 2a, \quad [c, b] = -2b, \quad [a, b] = c,$$

then one may define $e_1 = \frac{a+b}{2}$, $e_2 = \frac{a-b}{2}$, $e_3 = \frac{c}{2}$ to obtain

$$[e_1, e_2] = -e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = e_2.$$

None of the examples in (i)-(iii) of Riemannian manifolds of class $\mathcal{M}_\beta$ with $\beta = \frac{1}{2}$ are (locally) isometric one to the other. This fact is immediately read from the different values of K_2 (constant). Hence by Remarks C.10 and C.11, we see that for every $\beta \in \mathbb{R}$ there are non-isometric Riemannian manifolds of the same class $\mathcal{M}_\beta$.

C.3 Warped Products

Definition C.13 Let (M, g), (N, h) be Riemannian manifolds and $f \in C^\infty(M)$. Define a metric h_f on $M \times N$

$$h_f = \mathrm{pr}_1^*(g) + (f \circ \mathrm{pr}_1)^2 \mathrm{pr}_2^*(h),$$

where $\mathrm{pr}_1, \mathrm{pr}_2$ are projections onto the first and second factor of $M \times N$, respectively. Then the Riemannian manifold $(M \times N, h_f)$ is called a *warped product of* (M, g) *and* (N, h) with the *warping function* f. One may write $(M \times N, h_f)$ as $(M, g) \times_f (N, h)$ and h_f as $g \oplus_f h$ if there is a risk of ambiguity.

We are mainly interested in the case where $(M, g) = (I, s_1)$, where $I \subset \mathbb{R}$ is an open non-empty interval and s_1 is the standard Euclidean metric on $\mathbb{R}$. By convention, we write $\frac{\partial}{\partial r}$ for the natural positively directed unit (w.r.t. s_1) vector field on $\mathbb{R}$ and identify it in the canonical way as a vector field on the product $I \times N$ and notice that it is also a unit vector field w.r.t. h_f.

Since needed in section 5, we state (a local version of) the main result of [12] in 3-dimensional case. The general result allows one to detect Riemannian spaces which are locally warped products. In our setting we use it (in the below form) to detect when a 3-dimensional Riemannian manifold (M, g) is, around a given point, a warped product of the form $(I \times N, h_f)$, with $I \subset \mathbb{R}$, $f \in C^\infty(I)$, and (N, h) a 2-dimensional Riemannian manifold.

Theorem C.14 ([12]) Let (M, g) be a Riemannian manifold of dimension 3. Suppose that at every point $x_0 \in M$ there is an orthonormal frame E_1, E_2, E_3 defined in a neighbourhood of x_0 such that the connection table w.r.t. E_1, E_2, E_3 on this neighbourhood is of the form

$$\Gamma = \begin{pmatrix} 0 & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & 0 \end{pmatrix},$$

and moreover

$$X(\Gamma^1_{(1,2)}) = 0, \quad \forall X \in E_2^{\perp}.$$

Then there is a neighbourhood U of x, an interval $I \subset \mathbb{R}$, $f \in C^{\infty}(I)$ and a 2-dimensional Riemannian manifold (N, h) such that $(U, g|_U)$ is isometric to the warped product $(I \times N, h_f)$. If $F : (I \times N, h_f) \to (U, g|_U)$ is the isometry in question, then for all $(r, y) \in I \times N$,

$$\frac{f'(r)}{f(r)} = -\Gamma^1_{(1,2)}(F(r, y)), \quad F_* \frac{\partial}{\partial r}\Big|_{(r,y)} = E_2|_{\phi(r,y)}.$$

C.4 Technical propositions

Since we will be dealing frequently with orthonormal frames and connection coefficients, it is convenient to define the following concept.

Definition C.15 Let (M, g) be a 3-dimensional Riemannian manifold. If E_1, E_2, E_3 is an orthonormal frame of M defined on an open set U, then $\Gamma^j_{(i,k)} = g(\nabla_{E_j} E_i, E_k)$, we call the matrix

$$\Gamma = \begin{pmatrix} \Gamma^1_{(2,3)} & \Gamma^2_{(2,3)} & \Gamma^3_{(2,3)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & \Gamma^2_{(1,2)} & \Gamma^3_{(1,2)} \end{pmatrix},$$

the *connection table w.r.t.* E_1, E_2, E_3. To emphasize the frame, we may write $\Gamma = \Gamma_{(E_1,E_2,E_3)}$.

Remark C.16 (i) Since E_1, E_2, E_3 is orhonormal, one has $\Gamma^i_{(j,k)} = -\Gamma^i_{(k,j)}$ for all i, j, k. These relations mean that to know all the connection coefficients (of an orthonormal frame), it is enough to know exactly 9 of them. It is these 9 coefficients, that appear in the connection table.

(ii) Here it is important that the frame E_1, E_2, E_3 is ordered and hence one should speak of the connection table w.r.t. (E_1, E_2, E_3) (as in the notation $\Gamma = \Gamma_{(E_1,E_2,E_3)}$), but since we always list the frame in the correct order, there will be no room for confusion.

(iii) Notice that the above connection table could be written as $\Gamma = [(\Gamma^j_{\star i})^i_j]$, if one writes $\star 1 = (2, 3)$, $\star 2 = (3, 1)$ and $\star 3 = (1, 2)$ i.e.

$$\Gamma = \begin{pmatrix} \Gamma^1_{\star 1} & \Gamma^2_{\star 1} & \Gamma^3_{\star 1} \\ \Gamma^1_{\star 2} & \Gamma^2_{\star 2} & \Gamma^3_{\star 2} \\ \Gamma^1_{\star 3} & \Gamma^2_{\star 3} & \Gamma^3_{\star 3} \end{pmatrix}.$$

(iv) One should notice that usually the Christoffel symbols Γ^k_{ji} of a frame E_1, E_2, E_3 are defined by

$$\nabla_{E_j} E_i = \sum_{k=1}^{3} \Gamma^k_{ji} E_k.$$

The notation $\Gamma^j_{(i,k)}$ introduced above differs from this by

$$\Gamma^j_{(i,k)} = \Gamma^k_{ji}.$$

We find this notation convenient and we only use it when dealing with 3-dimensional manifolds.

Proposition C.17 Suppose (M, g) is a 3-dimensional Riemannian manifold and in some neighbourhood U of $x \in M$ there is an orthonormal frame E_1, E_2, E_3 defined on an open set U with respect to which the connection table is of the form

$$\Gamma = \begin{pmatrix} \Gamma^1_{(2,3)} & 0 & -\Gamma^1_{(1,2)} \\ \Gamma^1_{(3,1)} & \Gamma^2_{(3,1)} & \Gamma^3_{(3,1)} \\ \Gamma^1_{(1,2)} & 0 & \Gamma^1_{(2,3)} \end{pmatrix},$$

and $V(\Gamma^1_{(2,3)}) = 0$, $V(\Gamma^1_{(1,2)}) = 0$, for all $V \in E_2|_y^{\perp}$, $\quad y \in U$. Then the following are true:

(i) For every $y \in U$, $\star E_1|_y, \star E_2|_y, \star E_3|_y$ are eigenvectors of R with eigenvalues $-K(y), -K_2(y), -K(y)$, respectively (i.e. the eigenvalues of $\star E_1|_y$ and $\star E_3|_y$ coincide).

(ii) If $\Gamma^1_{(2,3)} \neq 0$ on U and if U is connected, it follows that on U the coefficient $\Gamma^1_{(2,3)}$ is constant, $\Gamma^1_{(1,2)} = 0$ and $K(y) = (\Gamma^1_{(2,3)})^2$ (constant). Hence $(U, g|_U)$ is of class $\mathcal{M}_\beta$, for $\beta = \Gamma^1_{(2,3)}$.

(iii) If $\Gamma^1_{(2,3)} = 0$ in the open set U, then every $y \in U$ has a neighbourhood $U' \subset U$ such that $(U', g|_{U'})$ is isometric to a warped product $(I \times N, h_f)$ where $I \subset \mathbb{R}$ is an open interval. Moreover, if $F : (I \times N, h_f) \to (U', g|_{U'})$ is the isometry in question, then, for every $(r, y) \in I \times N$,

$$\frac{f'(r)}{f(r)} = -\,\Gamma^1_{(1,2)}(F(r,y)), \quad F_* \frac{\partial}{\partial r}\Big|_{(r,y)} = E_2|_{F(r,y)}.$$

Moreover, one has

$$0 = -\,E_2(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)}, \tag{56}$$

$$-K = -\,E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2 - (\Gamma^1_{(2,3)})^2, \tag{57}$$

$$-K_2 = E_3(\Gamma^1_{(3,1)}) - E_1(\Gamma^3_{(3,1)}) + (\Gamma^1_{(3,1)})^2 + (\Gamma^3_{(3,1)})^2 - 2\Gamma^1_{(2,3)}\Gamma^2_{(3,1)} + (\Gamma^1_{(1,2)})^2 + (\Gamma^1_{(2,3)})^2. \tag{58}$$

Proof. (i) We begin by computing in the basis $\star E_1, \star E_2, \star E_3$ that

$$R(E_3 \wedge E_1) = \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix} \wedge \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} + \begin{pmatrix} E_3(\Gamma^1_{(2,3)}) \\ E_3(\Gamma^1_{(3,1)}) \\ E_3(\Gamma^1_{(1,2)}) \end{pmatrix} - \begin{pmatrix} -E_1(\Gamma^1_{(1,2)}) \\ E_1(\Gamma^3_{(3,1)}) \\ E_1(\Gamma^1_{(2,3)}) \end{pmatrix}$$
$$+ \Gamma^1_{(3,1)} \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} - 2\Gamma^1_{(2,3)} \begin{pmatrix} 0 \\ \Gamma^2_{(3,1)} \\ 0 \end{pmatrix} + \Gamma^3_{(3,1)} \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix} = \begin{pmatrix} 0 \\ -K_2 \\ 0 \end{pmatrix},$$

where we omitted the further computation of row 2 and wrote it simply as $-K_2$ and use the fact that $E_i(\Gamma^1_{(1,2)}) = 0$, $E_i(\Gamma^1_{(2,3)}) = 0$ for $i \in \{1, 3\}$. Thus $\star E_2|_y$ is an eigenvector of $R|_y$ for all $y \in U$. Since $R|_y$ is a symmetric linear map $\wedge^2 T|_y M$ to itself and $\star E_2|_y$ is an eigenvector for $R|_y$, we know that the other eigenvectors lie in $\star E_2|_y$, which is spanned by $\star E_1|_y, \star E_3|_y$. By rotating E_1, E_3 among themselves by a constant matrix, we may well assume that $\star E_1|_y, \star E_3|_y$ are eigenvectors of $R|_y$ corresponding to eigenvalues, say, $-K_1(y), -K_3(y)$. We want to show that $K_1(y) = K_3(y)$. Computing $R|_y(E_1 \wedge E_2)$ in the basis $\star E_1|_y, \star E_2|_y, \star E_3|_y$ gives (we write simply $\Gamma^i_{(j,k)}$ for $\Gamma^i_{(j,k)}(y)$ etc.)

$$\begin{pmatrix} 0 \\ 0 \\ -K_3(y) \end{pmatrix} = R|_y(\star E_3) = R|_y(E_1 \wedge E_2)$$
$$= \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \Gamma^2_{(3,1)} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ E_1(\Gamma^2_{(3,1)}) \\ 0 \end{pmatrix} - \begin{pmatrix} E_2(\Gamma^1_{(2,3)}) \\ E_2(\Gamma^1_{(3,1)}) \\ E_2(\Gamma^1_{(1,2)}) \end{pmatrix}$$
$$+ \Gamma^1_{(1,2)} \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} - (\Gamma^1_{(2,3)} + \Gamma^2_{(3,1)}) \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix}$$
$$= \begin{pmatrix} -E_2(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)} \\ E_1(\Gamma^2_{(3,1)}) - E_2(\Gamma^1_{(3,1)}) + \Gamma^1_{(1,2)}\Gamma^1_{(3,1)} - (\Gamma^1_{(2,3)} + \Gamma^2_{(3,1)})\Gamma^3_{(3,1)} \\ -E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2 - (\Gamma^1_{(2,3)})^2 \end{pmatrix},$$

from where $-K_3(y) = -E_2|_y(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)}(y))^2 - (\Gamma^1_{(2,3)}(y))^2$. Similarly, computing $R|_y(E_2 \wedge E_3)$ in the basis $\star E_1|_y, \star E_2|_y, \star E_3|_y$,

$$\begin{pmatrix} -K_1(y) \\ 0 \\ 0 \end{pmatrix} = R|_y(\star E_1) = R|_y(E_2 \wedge E_3)$$
$$= \begin{pmatrix} 0 \\ \Gamma^2_{(3,1)} \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix} + \begin{pmatrix} -E_2(\Gamma^1_{(1,2)}) \\ E_2(\Gamma^3_{(3,1)}) \\ E_2(\Gamma^1_{(2,3)}) \end{pmatrix} - \begin{pmatrix} 0 \\ E_3(\Gamma^2_{(3,1)}) \\ 0 \end{pmatrix}$$
$$- (\Gamma^2_{(3,1)} + \Gamma^1_{(2,3)}) \begin{pmatrix} \Gamma^1_{(2,3)} \\ \Gamma^1_{(3,1)} \\ \Gamma^1_{(1,2)} \end{pmatrix} - \Gamma^1_{(1,2)} \begin{pmatrix} -\Gamma^1_{(1,2)} \\ \Gamma^3_{(3,1)} \\ \Gamma^1_{(2,3)} \end{pmatrix}$$
$$= \begin{pmatrix} -E_2(\Gamma^1_{(1,2)}) - (\Gamma^1_{(2,3)})^2 + (\Gamma^1_{(1,2)})^2 \\ E_2(\Gamma^3_{(3,1)}) - E_3(\Gamma^2_{(3,1)}) - (\Gamma^2_{(3,1)} + \Gamma^1_{(2,3)})\Gamma^1_{(3,1)} - \Gamma^1_{(1,2)}\Gamma^3_{(3,1)} \\ E_2(\Gamma^1_{(2,3)}) - 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)}, \end{pmatrix}$$

leads us to $-K_1(y) = -E_2|_y(\Gamma^1_{(1,2)}) - (\Gamma^1_{(2,3)}(y))^2 + (\Gamma^1_{(1,2)}(y))^2$. By comparing to the result of the computations of $R|_y(E_1 \wedge E_2)$ and $R|_y(E_2 \wedge E_3)$ implies that $K_1(y) = K_3(y)$. In other words, if one writes $K(y)$ for this common value $K_1(y) = K_3(y)$, one sees that $E_2|_y^\perp$ is contained in the eigenspace of $R|_y$ corresponding to the eigenvalue $-K(y)$. This finishes the proof of (i).

(ii) Suppose now that $\Gamma^1_{(2,3)} \neq 0$ on an open connected subset U of $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}(O)$. Then since $E_1(\Gamma^1_{(2,3)}) = 0$, $E_3(\Gamma^1_{(2,3)}) = 0$ on U, one has, on U,

$$[E_3, E_1](\Gamma^1_{(2,3)}) = E_3(E_1(\Gamma^1_{(2,3)}))E_1(E_3(\Gamma^1_{(2,3)})) = 0.$$

On the other hand, $[E_3, E_1] = -\Gamma^1_{(3,1)}E_1 + 2\Gamma^1_{(2,3)}E_2 - \Gamma^3_{(3,1)}E_3$, so

$$\begin{aligned}
0 =& [E_3, E_1](\Gamma^1_{(2,3)}) \\
=& -\Gamma^1_{(3,1)}E_1(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(2,3)}E_2(\Gamma^1_{(2,3)}) - \Gamma^3_{(3,1)}E_3(\Gamma^1_{(2,3)}) = 2\Gamma^1_{(2,3)}E_2(\Gamma^1_{(2,3)}).
\end{aligned}$$

Since $\Gamma^1_{(2,3)} \neq 0$ everywhere on U, one has $E_2(\Gamma^1_{(2,3)}) = 0$ on U. Because E_1, E_2, E_3 span TM on U, we have that all the derivatives of $\Gamma^1_{(2,3)}$ vanish on U and thus it is constant. From the first row of the computation of $R(E_1 \wedge E_2)$ in the case (ii) above, one gets

$$0 = -E_2(\Gamma^1_{(2,3)}) + 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)} = 2\Gamma^1_{(1,2)}\Gamma^1_{(2,3)},$$

which implies $\Gamma^1_{(1,2)} = 0$ on U. Finally from the last row computation of $R(E_1 \wedge E_2)$ (recall that $K_1(y) = K_3(y) =: K(y)$), one gets

$$-K(y) = -E_2(\Gamma^1_{(1,2)}) + (\Gamma^1_{(1,2)})^2 - (\Gamma^1_{(2,3)})^2 = -(\Gamma^1_{(2,3)})^2.$$

This concludes the proof of (ii).

(iii) This case follows from Theorem C.14. □

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B Rolling Manifolds on Space Form

Joint work with Y. Chitour
(published in Ann. I. H. Poincaré - AN 29 (2012) 927-954)

Rolling Manifolds on Space Forms*

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April 24, 2012

Abstract

In this paper, we consider the rolling problem (R) without spinning nor slipping of a smooth connected oriented complete Riemannian manifold (M, g) onto a space form $(\hat{M}, \hat{g})$ of the same dimension $n \geq 2$. This amounts to study an n-dimensional distribution $\mathcal{D}_{\mathrm{R}}$, that we call the rolling distribution, and which is defined in terms of the Levi-Civita connections ∇^g and $\nabla^{\hat{g}}$. We then address the issue of the complete controllability of the control system associated to $\mathcal{D}_{\mathrm{R}}$. The key remark is that the state space Q carries the structure of a principal bundle compatible with $\mathcal{D}_{\mathrm{R}}$. It implies that the orbits obtained by rolling along loops of (M, g) become Lie subgroups of the structure group of $\pi_{Q,M}$. Moreover, these orbits can be realized as holonomy groups of either certain vector bundle connections ∇^{Rol}, called the rolling connections, when the curvature of the space form is non zero, or of an affine connection (in the sense of [16]) in the zero curvature case. As a consequence, we prove that the rolling (R) onto an Euclidean space is completely controllable if and only if the holonomy group of (M, g) is equal to $\mathrm{SO}(n)$. Moreover, when $(\hat{M}, \hat{g})$ has positive (constant) curvature we prove that, if the action of the holonomy group of ∇^{Rol} is not transitive, then (M, g) admits $(\hat{M}, \hat{g})$ as its universal covering. In addition, we show that, for n even and $n \geq 16$, the rolling problem (R) of (M, g) against the space form $(\hat{M}, \hat{g})$ of positive curvature $c > 0$, is completely controllable if and only if (M, g) is not of constant curvature c.

1 Introduction

In this paper, we study the rolling of a manifold over another one. Unless otherwise precised, manifolds are smooth, connected, oriented, of finite dimension $n \geq 2$, endowed with a Riemannian metric. The rolling is assumed to be without spinning (NS) or without spinning nor slipping (R). Here we only consider the rolling problem (R). When both manifolds are isometrically embedded into an Euclidean space, the rolling problem is classical in differential geometry (see [25]), through the notions of "development of a manifold" and "rolling maps". For instance, É. Cartan defines holonomy by rolling a manifold against its tangent space without spinning nor slipping (cf. [5, 7]). The most basic issue linked to the rolling problem (R) is that of *controllability* i.e., to determine, for two given points q_{init} and q_{final} in the state space Q, if there exists a curve γ so that the rolling (R) along γ steers the system from q_{init} to q_{final}. If

*The work of the first author is supported by the ANR project GCM, program "Blanche", (project number NT09_504490) and the DIGITEO-Région Ile-de-France project CONGEO. The work of the second author is supported by Finnish Academy of Science and Letters, KAUTE Foundation and l'Institut français de Finlande.

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this is the case for every points $q_{\rm init}$ and $q_{\rm final}$ in Q, then the rolling (R) is said to be *completely controllable*.

If the manifolds rolling on each other are two-dimensional, the controllability issue is well-understood thanks to the work of [2, 6, 8, 19, 1] especially. For instance, in the simply connected case, the rolling (R) is completely controllable if and only if the manifolds are not isometric. In the case where the manifolds are isometric, [2] also provides a description of the reachable sets in terms of isometries between the manifolds. In particular, these reachable sets are immersed submanifolds of Q of dimension either 2 or 5. In case the manifolds rolling on each other are isometric convex surfaces, [19] provides a beautiful description of a two dimensional reachable set: consider the initial configuration given by two (isometric) surfaces in contact so that one is the image of the other one by the symmetry with respect to the (common) tangent plane at the contact point. Then, this symmetry property (chirality) is preserved along the rolling (R). If the (isometric) convex surfaces are not spheres nor planes, the reachable set starting at a contact point where the Gaussian curvatures are distinct, is open (and thus of dimension 5).

After [2], the state space (Q) of the rolling problem (R) is given by

$$Q = \{A : T|_x M \to T|_{\hat{x}} \hat{M} \mid A \text{ o-isometry},\ x \in M,\ \hat{x} \in \hat{M}\},$$

where "o-isometry" means positively oriented isometry, (see [6, 20, 11] an alternative description). The set of admissible controls is equal to the set of absolutely continuous (a.c.) curves on M. We next construct an n-dimensional distribution $\mathcal{D}_{\rm R}$, that we call the rolling distribution, so that its tangent curves coincide with the admissible curves of $(\Sigma)_R$. A standard procedure in geometric control in order to address the controllability issue simply consists of studying the Lie algebra spanned by the vector fields tangent to $\mathcal{D}_{\rm R}$. More precisely, one tries to compute the dimension of the evaluation at every point $q \in Q$ of this Lie algebra. However, this strategy turns out to be delicate for the rolling problem, even if one of the manifolds is assumed to be the Euclidean space. Indeed, in that particular case, this amounts to determine the dimension of the holonomy group associated to the Levi-Civita connection of a Riemannian manifold (M, g), only from the infinitesimal information provided by the evaluation at any point x of the curvature tensor associated to ∇^g and its covariant derivatives of arbitrary order (cf. [10] for more details).

However, when one of the manifolds, let say $(\hat{M}, \hat{g})$, is a space form i.e., a simply connected complete Riemannian manifold of constant curvature, we prove, in Section 4, that there is a principal bundle structure on the bundle $\pi_{Q,M} : Q \to M$, which is compatible with the rolling distribution $\mathcal{D}_{\rm R}$. From this fundamental feature, we show how to address the complete controllability of the rolling problem (R) without resorting to any Lie bracket computation. Indeed, if $\hat{M}$ has zero curvature i.e., it is the Euclidean plane, we reduce the description of reachable sets to the study of an affine connection and its holonomy group, a subgroup of $\mathrm{SE}(n)$, in the sense of [16]. Then, we deduce that the rolling (R) is completely controllable if and only if the (Riemannian) holonomy group of ∇^g is equal to $\mathrm{SO}(n)$. This result is actually similar to Theorem IV.7.1, p. 193 and Theorem IV.7.2, p. 194 in [16].

In the case where $\hat{M}$ has non-zero constant curvature (up to a trivial reduction equal to 1 or -1), the description of reachable sets resumes to the study of a vector bundle connection ∇^{Rol} of the vector bundle $\pi_{TM\oplus\mathbb{R}} : TM \oplus \mathbb{R} \to M$ and its holonomy group H^{Rol}, which is a subgroup of $\mathrm{SO}(n+1)$ or $\mathrm{SO}(n,1)$ depending whether the curvature of $\hat{M}$ is equal to 1 or -1 respectively. We then prove that the rolling problem (R) is completely controllable if and only if H^{Rol} is equal to $\mathrm{SO}(n+1)$ or $\mathrm{SO}(n,1)$ respectively.

The structure of H^{Rol} is further investigated for the rolling onto an n-dimensional unit sphere S^n. We prove that if the action of H^{Rol} onto S^n is not transitive, then (M, g) admits the unit sphere as Riemmanian universal covering. This rigidity result can be seen as a de Rham type of result of global nature and we will provide in another paper ([9]) the details of the extension of de Rham decomposition theorem to the case of rolling on a space form of negative curvature.

Then by adapting to the classical argument of Simons [26] to our particular situation, we prove that for n even and $n \geq 16$, the rolling problem (R) of (M, g) against the space form $(\hat{M}, \hat{g})$ of positive curvature $c > 0$, is completely controllable if and only if (M, g) is not of constant curvature c. In that way, we recover some of the results of [15].

To conclude this introduction, we would like to propose some open problems. The first one deals with the rolling problem of two (locally) symmetric spaces. Indeed, the Lie algebraic structure of the rolling distribution does not involve the covariant derivatives of the curvature tensors on M and $\hat{M}$ (see [11]) and therefore its analysis turns out to be a purely algebraic question. Another question refers to the rolling onto a space of constant positive curvature, where the action of the rolling holonomy group is irreducible and transitive. One reasonably expects a list of possibilities similar to that of Berger. In addition, one may investigate the structure of the group of (local) symmetries associated to the rolling distribution, in particular when both manifolds M and $\hat{M}$ have constant curvature. Finally, what could be necessary conditions on M and $\hat{M}$ insuring that the rolling distribution is a principal bundle connection over $Q \to M$? Recall that we provide here a sufficient condition for that, namely that $\hat{M}$ has constant curvature.

2 Notations

For any sets A, B, C and $U \subset A \times B$ and any map $F : U \to C$, we write U_a and U^b for the sets defined by $\{b \in B \mid (a, b) \in U\}$ and $\{a \in A \mid (a, b) \in U\}$ respectively. Similarly, let $F_a : U_a \to C$ and $F^b : U^b \to C$ be defined by $F_a(b) := F(a, b)$ and $F^b(a) := F(a, b)$ respectively. For any sets $V_1, \ldots, V_n$ the map $\mathrm{pr}_i : V_1 \times \cdots \times V_n \to V_i$ denotes the projection onto the i-th factor.

In this paper, a smooth manifold is a finite-dimensional, second countable, Hausdorff manifold (see e.g. [18]). For any smooth map $\pi : E \to M$ between smooth manifolds E and M, the set $\pi^{-1}(\{x\}) =: \pi^{-1}(x)$ is called the π-fiber over x and it is sometimes denoted by $E|_x$, when π is clear from the context. The set of smooth sections of π is denoted by $\Gamma(\pi)$. The value $s(x)$ of a section s at x is usually denoted by $s|_x$. For a smooth map $\pi : E \to M$ and $y \in E$, let $V|_y(\pi)$ be the set of all $Y \in T|_y E$ such that $\pi_*(Y) = 0$. If π is a smooth bundle, the collection of spaces $V|_y(\pi)$, $y \in E$, defines a smooth submanifold $V(\pi)$ of $T(E)$ and the restriction $\pi_{T(E)} : T(E) \to E$ to $V(\pi)$ is denoted by $\pi_{V(\pi)}$. In this case $\pi_{V(\pi)}$ is a vector subbundle of $\pi_{T(E)}$ over E.

One uses $\mathrm{VF}(M)$ to denote the set of smooth vector fields on M. The flow of a vector field $Y \in \mathrm{VF}(M)$ is a smooth onto map $\Phi_Y : D \to M$ defined on an open subset D of $\mathbb{R} \times M$ containing $\{0\} \times M$.

For any maps $\gamma : [a, b] \to M$, $\omega : [c, d] \to M$ into M such that $\gamma(b) = \omega(c)$ we define

$$\omega \sqcup \gamma : [a, b + d - c] \to M; \quad (\omega \sqcup \gamma)(t) = \begin{cases} \gamma(t), & t \in [a, b] \\ \omega(t - b + c), & t \in [b, b + d - c]. \end{cases}$$

Also we write $\gamma^{-1} : [a, b] \to M; \gamma^{-1}(t) := \gamma(b + a - t)$. In the space of loops $[0, 1] \to M$ based at some given point x_0, one defines an operation "." of concatenation by $\omega.\gamma := (t \mapsto \omega(\frac{t}{2})) \sqcup (t \mapsto \gamma(\frac{t}{2}))$. For $y \in M$, we use $\Omega_y(M)$ to denote the set of all piecewise C^1-loops $[0, 1] \to M$ of M based at y.

A continuous map $\gamma : I \to M$ from a real compact interval I into a smooth manifold M is called *absolutely continuous*, or *a.c.* for short if, for every $t_0 \in I$, there is a smooth coordinate chart (ϕ, U) of M such that $\gamma(t_0) \in U$ and $\phi \circ \gamma|_{\gamma^{-1}(U)}$ is absolutely continuous.

Given a smooth distribution $\mathcal{D}$ on M, we call an absolutely continuous curve $\gamma : I \to M$, $I \subset \mathbb{R}$, *$\mathcal{D}$-admissible* if γ it is tangent to $\mathcal{D}$ almost everywhere (a.e.) i.e., if for almost all $t \in I$

it holds that $\dot{\gamma}(t) \in \mathcal{D}|_{\gamma(t)}$. For $x_0 \in M$, the endpoints of all the $\mathcal{D}$-admissible curves of M starting at x_0 form the set called *$\mathcal{D}$-orbit through* x_0 and denoted $\mathcal{O}_{\mathcal{D}}(x_0)$. More precisely,

$$\mathcal{O}_{\mathcal{D}}(x_0) = \{\gamma(1) \mid \gamma : [0,1] \to M,\ \mathcal{D}\text{−admissible},\ \gamma(0) = x_0\}. \tag{1}$$

By the Orbit Theorem (see [3]), it follows that $\mathcal{O}_{\mathcal{D}}(x_0)$ is an immersed smooth submanifold of M containing x_0. It is also known that one may restrict to piecewise smooth curves in the description of the orbit i.e., the curves γ in (1) can be taken piecewise smooth.

Let $\pi : E \to M$ be a vector bundle and $\nabla : \mathrm{VF}(M) \times \Gamma(\pi) \to \Gamma(\pi)$ a linear connection on π. As is standard, we write for $X \in \mathrm{VF}(M)$, $s \in \Gamma(\pi)$ the value of ∇ as $\nabla_X s \in \Gamma(\pi)$. A parallel transport of $s_0 \in E|_{x_0}$ along an a.c. path $\gamma : [a,b] \to M$ from $\gamma(a) = x_0$ to $\gamma(b)$ is written as $(P^\nabla)^b_a(\gamma)s_0$. The parallel transport map

$$(P^\nabla)^b_a(\gamma) : E|_{\gamma(a)} \to E|_{\gamma(b)} \tag{2}$$

is a linear isomorphism and one also writes $(P^\nabla)^a_b(\gamma) := (P^\nabla)^b_a(\gamma^{-1}) = (P^\nabla)^b_a(\gamma)^{-1}$. The holonomy group of ∇ at x_0 is defined to be the subgroup $H^\nabla|_{x_0}$ of $\mathrm{GL}(E|_{x_0})$ given by

$$H^\nabla|_{x_0} = \{(P^\nabla)^1_0(\gamma) \mid \gamma \in \Omega_{x_0}(M)\}.$$

One writes R^∇ for the curvature tensor of ∇ and if the connection ∇ is clear from the context, one simply writes $P = P^\nabla$ and $R = R^\nabla$ for the parallel transport operator and the curvature operator, respectively. Finally, the Levi-Civita connection of a Riemannian manifold (N,h) is written as ∇^h or simply ∇ when h is clear from the context.

We use $\mathrm{Iso}(N,h)$ to denote the group of isometries of a Riemannian manifold (N,h). The isometries respect parallel transport in the sense that for any absolutely continuous $\gamma : [a,b] \to N$ and $F \in \mathrm{Iso}(N,h)$ one has (cf. [24], p. 41, Eq. (3.5))

$$F_*|_{\gamma(t)} \circ (P^{\nabla^h})^t_a(\gamma) = (P^{\nabla^h})^t_a(F \circ \gamma) \circ F_*|_{\gamma(a)}. \tag{3}$$

The following result is standard.

Proposition 2.1 (cf. [16], Chapter IV, Theorem 4.1) Let (N,h) be a Riemannian manifold and for any absolutely continuous $\gamma : [0,1] \to M$, $\gamma(0) = y_0$, define

$$\Lambda^{\nabla^h}_{y_0}(\gamma)(t) = \int_0^t (P^{\nabla^h})^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s \in T|_{y_0}N, \quad t \in [0,1].$$

Then the map $\Lambda^{\nabla^h}_{y_0} : \gamma \mapsto \Lambda^{\nabla^h}_{y_0}(\gamma)(\cdot)$ is an injection from the set of absolutely continuous curves $[0,1] \to N$ starting at y_0 onto an open subset of the Banach space of absolutely continuous curves $[0,1] \to T|_{y_0}N$ starting at 0. Moreover, the map $\Lambda^{\nabla^h}_{y_0}$ is a bijection onto the latter Banach space if (and only if) (N,h) is a complete Riemannian manifold.

3 State Space and Distributions

3.1 State Space

3.1.1 Definition of the state space

After [2], [3] we make the following definition.

Definition 3.1 The *state space* $Q = Q(M, \hat{M})$ for the rolling of two n-dimensional *connected, oriented* smooth Riemannian manifolds $(M, g), (\hat{M}, \hat{g})$ is defined as

$$Q = \{A : T|_x M \to T|_{\hat{x}} \hat{M} \mid A \text{ o-isometry},\ x \in M,\ \hat{x} \in \hat{M}\},$$

with "o-isometry" means "orientation preserving isometry": if $(X_i)_{i=1}^n$ is a pos. oriented g-orthonormal frame of M at x then $(AX_i)_{i=1}^n$ is a pos. oriented $\hat{g}$-orthonormal frame of $\hat{M}$ at $\hat{x}$.

The linear space of $\mathbb{R}$-linear map $A : T|_x M \to T|_{\hat{x}} \hat{M}$ is canonically isomorphic to the tensor product $T^*|_x M \otimes T|_{\hat{x}} \hat{M}$. We write

$$T^*M \otimes T\hat{M} = \bigcup_{(x,\hat{x}) \in M \times \hat{M}} T^*|_x M \otimes T|_{\hat{x}} \hat{M}.$$

and if a point $A \in T^*M \otimes T\hat{M}$ belongs to $T^*|_x M \otimes T|_{\hat{x}} \hat{M}$, we usually write it as $q = (x, \hat{x}; A)$. With projection $\pi_{T^*M \otimes T\hat{M}} : T^*M \otimes T\hat{M} \to M \times \hat{M}$; $(x, \hat{x}; A) \mapsto (x, \hat{x})$, the space $T^*M \otimes T\hat{M}$ becomes a vector bundle over $M \times \hat{M}$ of rank n^2 and $\pi_Q := \pi_{T^*M \otimes T\hat{M}}|_Q : Q \to M \times \hat{M}$ is a smooth subbundle of rank $n(n-1)/2$ with fibers diffeomorphic to $\mathrm{SO}(n)$.

Remark 3.2 Let $q = (x, \hat{x}; A) \in Q$ and $B \in (T^*M \otimes T\hat{M})|_{(x,\hat{x})}$. Then $\nu(B)|_q \in V|_q(\pi_{T^*M \otimes T\hat{M}})$ is tangent to Q (i.e., is an element of $V|_q(\pi_Q)$) if and only if $\hat{g}(AX, BY) + \hat{g}(BX, AY) = 0$ for all $X, Y \in T|_x M$. This latter condition can be stated equivalently as $B \in A(\mathfrak{so}(T|_x M))$, i.e. $V|_{(x,\hat{x};A)}(\pi_Q)$ is naturally $\mathbb{R}$-linearly isomorphic to $A(\mathfrak{so}(T|_x M))$.

3.2 Distribution and the Control Problems

3.2.1 The Rolling Distribution $\mathcal{D}_{\mathrm{R}}$

In this section, using the subsequent lift operation, we build a smooth distribution $\mathcal{D}_{\mathrm{R}}$ on the spaces Q and $T^*M \otimes T\hat{M}$ whose tangent curves are the solutions of (8). For the next definition, we use the fact that if $A \in Q$, then $P_0^t(\hat{\gamma}) \circ A \circ P_t^0(\gamma) \in Q$ for all t where $\gamma, \hat{\gamma}$ are any smooth curves in $M, \hat{M}$ respectively.

Definition 3.3 For $q = (x, \hat{x}; A) \in Q$ and $X \in T|_x M$ we define a vector $\mathscr{L}_{\mathrm{R}}(X)|_q \in T|_q Q$ as

$$\mathscr{L}_{\mathrm{R}}(X)|_q = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (P_0^t(\hat{\gamma}) \circ A \circ P_t^0(\gamma)) \tag{4}$$

where $\gamma, \hat{\gamma}$ are any smooth curves in $M, \hat{M}$ respectively such that $\dot{\gamma}(0) = X$ and $\dot{\hat{\gamma}}(0) = AX$.

Remark 3.4 The definition of $\mathscr{L}_{\mathrm{R}}(X)$ as given above is independent of the choice of $\gamma, \hat{\gamma}$ such that the satisfy $\dot{\gamma}(0) = X$, $\dot{\hat{\gamma}}(0) = AX$.

This map naturally induces $\mathscr{L}_{\mathrm{R}} : \mathrm{VF}(M) \to \mathrm{VF}(Q)$ as follows. For $X \in \mathrm{VF}(M)$ we define $\mathscr{L}_{\mathrm{R}}(X)$, the *rolling lifted* vector field associated to X, by

$$\begin{aligned} \mathscr{L}_{\mathrm{R}}(X) : Q &\to TQ, \\ q &\mapsto \mathscr{L}_{\mathrm{R}}(X)|_q. \end{aligned}$$

The rolling lift map $\mathscr{L}_{\mathrm{R}}$ allows one to construct a distribution on Q of rank n as follows.

Definition 3.5 The *rolling distribution* $\mathcal{D}_{\mathrm{R}}$ on Q is the n-dim. smooth distribution defined by

$$\forall q = (x, \hat{x}; A) \in Q, \qquad \mathcal{D}_{\mathrm{R}}|_q = \mathscr{L}_{\mathrm{R}}(T|_x M)|_q. \tag{5}$$

One defines $\pi_{Q,M} := \mathrm{pr}_1 \circ \pi_Q : Q \to M$.

Remark 3.6 The map $\pi_{Q,M} : Q \to M$ is a bundle: if $F = (X_i)_{i=1}^n$ is a local oriented orthonormal frame of M defined on an open set U, the local trivialization of $\pi_{Q,M}$ induced by F is

$$\tau_F : \pi_{Q,M}^{-1}(U) \to U \times F_{\mathrm{OON}}(\hat{M}); \quad \tau_F(x, \hat{x}; A) = (x, (AX_i|_x)_{i=1}^n),$$

is a diffeomorphism, where $F_{\mathrm{OON}}(\hat{M})$ is the bundle of all oriented orthonormal frames on $\hat{M}$.

The differential $(\pi_{Q,M})_*$ maps each $\mathcal{D}_{\mathrm{R}}|_q$, $q = (x, \hat{x}; A) \in Q$, isomorphically onto $T|_x M$, implying the local existence of rolling curves described in the following proposition (c.f. [10]).

Proposition 3.7 (i) For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and a. c. $\gamma : [0, a] \to M$, $a > 0$, such that $\gamma(0) = x_0$, there exists a unique a. c. $q : [0, a'] \to Q$, $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, with $0 < a' \leq a$ (and a' maximal with the latter property), which is tangent to $\mathcal{D}_{\mathrm{R}}$ a.e. and $q(0) = q_0$. We denote this unique curve q by

$$t \mapsto q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = (\gamma(t), \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t); A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t)),$$

and refer to it as the *rolling curve* with initial conditions (γ, q_0) or *along* γ *with initial position* q_0. In the case that $\hat{M}$ is a complete manifold one has $a' = a$.

Conversely, any absolutely continuous curve $q : [0, a] \to Q$, which is a.e. tangent to $\mathcal{D}_{\mathrm{R}}$, is a rolling curve along $\gamma := \pi_{Q,M} \circ q$ i.e., has the form $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q(0))$.

(ii) Writing $\Lambda_{x_0} = \Lambda^{\nabla}_{x_0}$ and $\hat{\Lambda}_{\hat{x}_0} = \hat{\Lambda}^{\hat{\nabla}}_{\hat{x}_0}$ (see Proposition 2.1), then, for any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and a.c. curve γ starting from x_0, the corresponding rolling curve is given by

$$q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = (\gamma(t), \hat{\Lambda}^{-1}_{\hat{x}_0}(A_0 \circ \Lambda_{x_0}(\gamma))(t); P_0^t\big(\hat{\Lambda}^{-1}_{\hat{x}_0}(A_0 \circ \Lambda_{x_0}(\gamma))\big) \circ A_0 \circ P_t^0(\gamma)\big). \tag{6}$$

(iii) Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, $X \in T|_{x_0} M$ and $\gamma : [0, a] \to M$; $\gamma(t) = \exp_{x_0}(tX)$, a geodesic of (M, g) with $\gamma(0) = x_0$, $\dot{\gamma}(0) = X$. The rolling curve $q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0) = (\gamma, \hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0); A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)) : [0, a'] \to Q$, $0 < a' \leq a$, along γ with initial position q_0 is given by

$$\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = \widehat{\exp}_{\hat{x}_0}(tA_0 X), \quad A_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(t) = P_0^t(\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)) \circ A_0 \circ P_t^0(\gamma),$$

where $\widehat{\exp}$ is the exponential mapping of $(\hat{M}, \hat{g})$. Moreover, $a' = a$ if $\hat{M}$ is complete.

(iv) If $\gamma : [a, b] \to M$ and $\omega : [c, d] \to M$ are two a.c. curves with $\gamma(b) = \omega(c)$ and $q_0 \in Q$, then

$$q_{\mathcal{D}_{\mathrm{R}}}(\omega \sqcup \gamma, q_0) = q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(b)) \sqcup q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0). \tag{7}$$

On the group $\Omega_{x_0}(M)$ of piecewise differentiable loops of M based at x_0 one has

$$q_{\mathcal{D}_{\mathrm{R}}}(\omega.\gamma, q_0) = q_{\mathcal{D}_{\mathrm{R}}}(\omega, q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1)).q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0),$$

where $\gamma, \omega \in \Omega_{x_0}(M)$.

Remark 3.8 The curve $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t)) \in Q$, $t \in [a, b]$ is a rolling curve if and only if it is an admissible curve of the following driftless control affine system

$$(\Sigma)_R \quad \begin{cases} \dot{\gamma}(t) = u(t), \\ \dot{\hat{\gamma}}(t) = A(t)u(t), \\ \overline{\nabla}_{(u(t), A(t)u(t))} A(t) = 0, \end{cases} \qquad \text{for a.e. } t \in [a, b]. \tag{8}$$

where $\overline{\nabla}$ is the vector bundle connection on $\pi_{T^*M\otimes T\hat{M}}$ canonically induced by $\nabla, \hat{\nabla}$ and the control u belongs to $\mathcal{U}(M)$, the set of measurable TM-valued functions u defined on some interval $I = [a, b]$ such that there exists a.c. $y : [a, b] \to M$ verifying $u = \dot{y}$ a.e. on $[a, b]$. We can write the above system as

$$\begin{cases} \dot{\hat{\gamma}}(t) = A(t)\dot{\gamma}(t), \\ \overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A(t) = 0 \end{cases}$$

where γ is a.c. In the model of rolling of a Riemannian manifold (M, g) against another one $(\hat{M}, \hat{g})$, the first (resp. second) equation models the so-called *no-slipping condition* (resp. *no-spinning condition*). A complete argument for the above remark is provided in [10].

3.3 Global properties of a $\mathcal{D}_{\rm R}$-orbit

The next proposition describes on one hand the symmetry of the rolling problem with respect to (M, g) and $(\hat{M}, \hat{g})$ and on the second hand that each $\mathcal{D}_{\rm R}$-orbit is a smooth bundle over M. Proofs are omitted (cf. [10]).

Proposition 3.9 (i) Let $\widehat{\mathcal{D}_{\rm R}}$ be the rolling distribution in $\hat{Q} := Q(\hat{M}, M)$. Then the map $\iota : Q \to \hat{Q};\quad \iota(x, \hat{x}; A) = (\hat{x}, x; A^{-1})$ is a diffeomorphism of Q onto $\hat{Q}$ and $\iota_*\mathcal{D}_{\rm R} = \widehat{\mathcal{D}_{\rm R}}$. In particular, $\iota(\mathcal{O}_{\mathcal{D}_{\rm R}}(q)) = \mathcal{O}_{\widehat{\mathcal{D}_{\rm R}}}(\iota(q))$.

(ii) Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and suppose that $\hat{M}$ is complete. Then $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0), M} := \pi_{Q,M}|_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)} : \mathcal{O}_{\mathcal{D}_{\rm R}}(q_0) \to M$, is a smooth subbundle of $\pi_{Q,M}$.

Proposition 3.10 For any Riemannian isometries $F \in \mathrm{Iso}(M, g)$ and $\hat{F} \in \mathrm{Iso}(\hat{M}, \hat{g})$ of (M, g), $(\hat{M}, \hat{g})$ respectively, one defines smooth free right and left actions of $\mathrm{Iso}(M, g)$, $\mathrm{Iso}(\hat{M}, \hat{g})$ on Q by

$$q_0 \cdot F := (F^{-1}(x_0), \hat{x}_0; A_0 \circ F_*|_{F^{-1}(x_0)}), \quad \hat{F} \cdot q_0 := (x_0, \hat{F}(\hat{x}_0); \hat{F}_*|_{\hat{x}_0} \circ A_0),$$

where $q_0 = (x_0, \hat{x}_0; A_0) \in Q$. Set $\hat{F} \cdot q_0 \cdot F := (\hat{F} \cdot q_0) \cdot F = \hat{F} \cdot (q_0 \cdot F)$. For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, a.c. $\gamma : [0, 1] \to M$, $\gamma(0) = x_0$, and $F \in \mathrm{Iso}(M, g)$, $\hat{F} \in \mathrm{Iso}(\hat{M}, \hat{g})$, one has,

$$\hat{F} \cdot q_{\mathcal{D}_{\rm R}}(\gamma, q_0)(t) \cdot F = q_{\mathcal{D}_{\rm R}}(F^{-1} \circ \gamma, \hat{F} \cdot q_0 \cdot F)(t), \tag{9}$$

for all $t \in [0, 1]$ where $q_{\mathcal{D}_{\rm R}}(\gamma, q_0)(t)$ is defined. In particular, $\hat{F} \cdot \mathcal{O}_{\mathcal{D}_{\rm R}}(q_0) \cdot F = \mathcal{O}_{\mathcal{D}_{\rm R}}(\hat{F} \cdot q_0 \cdot F)$.

Proof. The fact that the group actions are well defined is clear and the smoothness of these actions can be proven by writing out the Lie-group structures of the isometry groups (using e.g. Lemma III.6.4 in [24]). If $q_0 \cdot F = q_0 \cdot F'$ for some $F, F' \in \mathrm{Iso}(M, g)$ and $q_0 \in Q$, then $F^{-1}(x_0) = F'^{-1}(x_0)$, $F_*|_{x_0} = F'_*|_{x_0}$ and hence $F = F'$ since M is connected (see [24], p. 43). This proves the freeness of the right $\mathrm{Iso}(M, g)$-action. The same argument proves the freeness of the left $\mathrm{Iso}(\hat{M}, \hat{g})$-action.

Finally, Eq. (9) follows from a simple application of Eq. (3). Indeed, we first recall the rolling curve $q_{\mathcal{D}_{\rm R}}(\gamma, q_0) = (\gamma, \hat{\gamma}_{\mathcal{D}_{\rm R}}(\gamma, q_0); A_{\mathcal{D}_{\rm R}}(\gamma, q_0))$ satisfies

$$\begin{aligned} &P_t^0(\hat{\gamma}_{\mathcal{D}_{\rm R}}(\gamma, q_0))\dot{\hat{\gamma}}_{\mathcal{D}_{\rm R}}(\gamma, q_0)(t) = A_0 P_t^0(\gamma)\dot{\gamma}(t), \\ &A_{\mathcal{D}_{\rm R}}(\gamma, q_0)(t) = P_0^t(\hat{\gamma}_{\mathcal{D}_{\rm R}}(\gamma, q_0)) \circ A_0 \circ P_t^0(\gamma). \end{aligned}$$

First, by using (3), we get

$$\begin{aligned} &P_t^0(\hat{F} \circ \hat{\gamma}_{\mathcal{D}_{\rm R}}(\gamma, q_0))\frac{\mathrm{d}}{\mathrm{d}t}(\hat{F} \circ \hat{\gamma}_{\mathcal{D}_{\rm R}}(\gamma, q_0))(t) = \hat{F}_* P_t^0(\hat{\gamma}_{\mathcal{D}_{\rm R}}(\gamma, q_0))\hat{F}_*^{-1}(\hat{F}_*\dot{\hat{\gamma}}_{\mathcal{D}_{\rm R}}(\gamma, q_0)(t)) \\ =&\hat{F}_* A_0 P_t^0(\gamma)\dot{\gamma}(t) = (\hat{F}_* A_0 F_*)(F_*^{-1} P_t^0(\gamma) F_*)F_*^{-1}\dot{\gamma}(t) = (\hat{F}_* A_0 F_*)P_t^0(F^{-1} \circ \gamma)\frac{\mathrm{d}}{\mathrm{d}t}(F^{-1} \circ \gamma)(t), \end{aligned}$$

and since by definition one has

$$P_t^0(\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F))\dot{\hat{\gamma}}_{\mathcal{D}_{\mathrm{R}}}(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F)=(\hat{F}_*A_0F_*)P_t^0(F^{-1}\circ\gamma)\frac{\mathrm{d}}{\mathrm{d}t}(F^{-1}\circ\gamma)(t),$$

the uniqueness of solutions of a system of ODEs gives that $\hat{F}\circ\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0)=\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F)$. Hence (9) is a consequence of the following

$$\begin{aligned}
&\hat{F}_*A_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0)F_*=\hat{F}_*(P_0^t(\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0))\circ A_0\circ P_t^0(\gamma))F_*\\
=&P_0^t(\hat{F}\circ\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0))\circ(\hat{F}_*A_0F_*)\circ P_t^0(F^{-1}\circ\gamma)\\
=&P_0^t(\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F))\circ(\hat{F}_*A_0F_*)\circ P_t^0(F^{-1}\circ\gamma)=A_{\mathcal{D}_{\mathrm{R}}}(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F).
\end{aligned}$$

□

The following proposition and its corollary are given without their proofs.

Proposition 3.11 Let $\pi_1:(M_1,g_1)\to(M,g)$ and $\hat{\pi}:(\hat{M}_1,\hat{g}_1)\to(\hat{M},\hat{g})$ be Riemannian coverings. Write $Q_1=Q(M_1,\hat{M}_1)$ and $(\mathcal{D}_{\mathrm{R}})_1$ for the rolling distribution in Q_1. Then the map $\Pi:Q_1\to Q;\quad \Pi(x_1,\hat{x}_1;A_1)=\big(\pi(x_1),\hat{\pi}(\hat{x}_1);\hat{\pi}_*|_{\hat{x}_1}\circ A_1\circ(\pi_*|_{x_1})^{-1}\big)$ is a covering map of Q_1 over Q and $\Pi_*(\mathcal{D}_{\mathrm{R}})_1=\mathcal{D}_{\mathrm{R}}$. Moreover, for every $q_1\in Q_1$ the restriction onto $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)$ of Π is a covering map $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)\to\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$. Then, for every $q_1\in Q_1$, $\Pi(\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1))=\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))$ and one has $\mathcal{O}_{(\mathcal{D}_{\mathrm{R}})_1}(q_1)=Q_1$ if and only if $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(\Pi(q_1))=Q$.

As an immediate corollary of the above proposition, we obtain the following result regarding the complete controllability of $(\mathcal{D}_{\mathrm{R}})$.

Corollary 3.12 Let $\pi_1:(M_1,g_1)\to(M,g)$ and $\hat{\pi}:(\hat{M}_1,\hat{g}_1)\to(\hat{M},\hat{g})$ be Riemannian coverings. Write $Q=Q(M,\hat{M})$, $\mathcal{D}_{\mathrm{R}}$ and $Q_1=Q(M_1,\hat{M}_1)$, $(\mathcal{D}_{\mathrm{R}})_1$ respectively for the state space and for the rolling distribution in the respective state space. Then the control system associated to $\mathcal{D}_{\mathrm{R}}$ is completely controllable if and only if the control system associated to $(\mathcal{D}_{\mathrm{R}})_1$ is completely controllable. As a consequence, when one addresses the complete controllability issue for the rolling distribution $\mathcal{D}_{\mathrm{R}}$, one can assume with no loss of generality that both manifolds M and $\hat{M}$ are simply connected.

4 Rolling Against a Space Form

For the rest of the paper we assume that $(\hat{M},\hat{g})$ is a space form i.e., a simply connected complete Riemannian manifold of constant curvature. The possible cases are: (i) Euclidean space with Euclidean metric (zero curvature), (ii) Sphere (positive curvature) and (iii) Hyperbolic space (negative curvature), cf. e.g. [24].

We first reduce the original control problem to the following one: fix (any) $x_0\in M$ and consider rolling of M along loops $\gamma\in\Omega_{x_0}(M)$, one obtains a control problem whose state space is the fiber $\pi_{Q,M}^{-1}(x_0)$ and the reachable sets are $\pi_{Q,M}^{-1}\cap\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, where $q_0\in\pi_{Q,M}^{-1}(x_0)$. It is then trivial to see that complete controllability of the original problem is equivalent to the complete controllability of the reduced rolling problem. Note that this fact holds true for the general rolling problem of one Riemannian manifold against another one.

On the other hand, the rolling problem against a space form of constant curvature $c\in\mathbb{R}$ actually presents a fundamental feature which turns out to be the crucial ingredient to address the controllability issue. We next prove that on the bundle $\pi_{Q,M}:Q\to M$ one can define a principal bundle structure that preserves the rolling distribution $\mathcal{D}_{\mathrm{R}}$. As a consequence, the reachable sets $\pi_{Q,M}^{-1}(x_0)\cap\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ become Lie subgroups of the structure group of $\pi_{Q,M}$. We will prove that these orbits in fact can be realized as holonomy groups of certain vector bundle connections if $c \neq 0$ and as a holonomy group of an affine connection (in the sense of [16]). Therefore the original problem of complete controllability reduces to the study of appropriate connections.

4.1 Orbit Structure

We first by recalling standard results on space forms. Following section V.3 of [16], we define the n-dimensional space form $\hat{M}_{n;c}$ of curvature $c \neq 0$ as a subset of $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$, given by

$$\hat{M}_{n;c} := \big\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 + c^{-1}x_{n+1}^2 = c^{-1},\ x_{n+1} + \frac{c}{|c|} \geq 0\big\}.$$

Equip $\hat{M}_{n;c}$ with a Riemannian metric $\hat{g}_{n;c}$ defined as the restriction to $\hat{M}_{n;c}$ of the non-degenerate symmetric $(0,2)$-tensor $s_{n;c} := (\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_n)^2 + c^{-1}(\mathrm{d}x_{n+1})^2$. The condition $x_{n+1} + \frac{c}{|c|} \geq 0$ in the definition $\hat{M}_{n;c}$ guarantees that $\hat{M}_{n;c}$ is connected also when $c < 0$.

Let $G_c(n)$ be the Lie group of linear maps $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ leaving invariant the bilinear form

$$\langle x, y \rangle_{n;c} := \sum_{i=1}^{n} x_i y_i + c^{-1}x_{n+1}y_{n+1},$$

for $x = (x_1, \ldots, x_{n+1})$, $y = (y_1, \ldots, y_{n+1})$ and having determinant $+1$. In other words, a linear map $B : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ belongs to $G_c(n)$ if and only if $\det(B) = +1$ and $\langle Bx, By \rangle_{n;c} = \langle x, y \rangle_{n;c}$, $\forall x, y \in \mathbb{R}^{n+1}$, or, equivalently, $B^T I_{n;c} B = I_{n;c}$, $\det(B) = +1$, where $I_{n;c} = \mathrm{diag}(1, 1, \ldots, 1, c^{-1})$. In particular, $G_1(n) = \mathrm{SO}(n+1)$ and $G_{-1}(n) = \mathrm{SO}(n,1)$. The Lie algebra of the Lie group $G_c(n)$ will be denoted by $\mathfrak{g}_c(n)$. Notice that an $(n+1) \times (n+1)$ real matrix B belongs to $\mathfrak{g}_c(n)$ if and only if $B^T I_{n;c} + I_{n;c} B = 0$, where $I_{n;c}$ was introduced above.

Sometimes we identify the form $s_{n;c}$ on $\mathbb{R}^{n+1}$ with $\langle \cdot, \cdot \rangle_{n;c}$ using the canonical identification of the tangent spaces $T|_v\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$. Notice that if $\hat{x} \in \hat{M}_{n;c}$ and $V \in T|_{\hat{x}}\mathbb{R}^{n+1}$, then $V \in T|_{\hat{x}}\hat{M}_{n;c}$ if and only if $s_{n;c}(V, \hat{x}) = 0$.

If $c = 0$, the space form $(\hat{M}_{n;0}, \hat{g}_{n;0})$ is simply equal to $\mathbb{R}^n$ with the Euclidean metric, $G_n(0)$ is set to be the group $\mathrm{SE}(n) := \mathrm{SE}(\mathbb{R}^n)$, the special Euclidean group of $(\hat{M}_{n;0}, \hat{g}_{n;0})$. Recall that $\mathrm{SE}(n)$ is equal to $\mathbb{R}^n \times \mathrm{SO}(n)$ as a set, and is equipped with the group operation $\star$ given by

$$(v, L) \star (u, K) := (Lu + v, L \circ K).$$

The natural action, also written as $\star$, of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ is given by

$$(u, K) \star v := Kv + u, \quad (u, K) \in \mathrm{SO}(V),\ v \in V.$$

Finally recall that, with this notation, the isometry group of $(\hat{M}_{n;c}, \hat{g}_{n;c})$ is equal to $G_c(n)$ for all $c \in \mathbb{R}$ (cf. [16]) as explicitly recalled in the next proof. From now on we set $(\hat{M}, \hat{g}) = (\hat{M}_{n;c}, \hat{g}_{n;c})$ for $c \in \mathbb{R}$. In the next proposition we detail the principal bundle structure of $\pi_{Q,M}$.

Proposition 4.1 (i) The bundle $\pi_{Q,M} : Q \to M$ is a principal $G_c(n)$-bundle with a left action $\mu : G_c(n) \times Q \to Q$ defined for every $q = (x, \hat{x}; A)$ by

$$\begin{aligned} \mu((\hat{y}, C), q) =& (x, C\hat{x} + \hat{y}; C \circ A), && \text{if } c = 0 \\ \mu(B, q) =& (x, B\hat{x}; B \circ A), && \text{if } c \neq 0 \end{aligned}$$

Moreover, the action μ preserves the distribution $\mathcal{D}_{\mathrm{R}}$ i.e., for any $q \in Q$ and $B \in G_c(n)$, $(\mu_B)_* \mathcal{D}_{\mathrm{R}}|_q = \mathcal{D}_{\mathrm{R}}|_{\mu(B,q)}$ where $\mu_B : Q \to Q;\ q \mapsto \mu(B, q)$.

(ii) For any given $q = (x, \hat{x}; A) \in Q$ there is a unique subgroup $\mathcal{H}_q$ of $G_c(n)$, called the holonomy group of $\mathcal{D}_{\mathrm{R}}$, such that

$$\mu(\mathcal{H}_q \times \{q\}) = \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \cap \pi_{Q,M}^{-1}(x).$$

Also, if $q' = (x, \hat{x}'; A') \in Q$ is in the same $\pi_{Q,M}$-fiber as q, then $\mathcal{H}_q$ and $\mathcal{H}_{q'}$ are conjugate in $G_c(n)$ and all conjugacy classes of $\mathcal{H}_q$ in $G_c(n)$ are of the form $\mathcal{H}_{q'}$. This conjugacy class will be denoted by $\mathcal{H}$. Moreover, $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q),M} : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q) \to M$ is a principal $\mathcal{H}$-bundle over M.

Proof. (i) We begin by showing that if $B \in G_c(n)$, then $\mu(B, q) \in Q$. Let $X \in T|_x$, If $c = 0$, then $B = (\hat{y}, C) \in \mathrm{SE}(n) = \mathbb{R}^n \times \mathrm{SO}(n)$ and

$$\|\mu(B,q)X\|_{\hat{g}_{n;0}} = \|CAX\|_{\hat{g}_{n;0}} = \|AX\|_{\hat{g}_{n;0}} = \|X\|_g$$

while if $c \neq 0$,

$$\|\mu(B,q)X\|_{\hat{g}_{n;c}} = \|BAX\|_{\hat{g}_{n;c}} = \|AX\|_{\hat{g}_{n;c}} = \|X\|_g.$$

Since $G_c(n)$ is connected for every $c \in \mathbb{R}$, it follows that $\mu(B, q) = (x, \hat{z}; A')$ viewed as a map $T|_x M \to T_{\hat{z}} \hat{M}_{n;c}$ is also orientation preserving and therefore indeed $\mu(B, q) \in Q$.

Clearly μ is smooth, satisfies the group action property and the action is free. We show that μ-action is transitive and proper, implying that $\pi_{Q,M}$ endowed with $G_c(n)$ action μ becomes a principal bundle.

Let $q = (x, \hat{x}; A), q' = (x, \hat{x}', A') \in \pi_{Q,M}^{-1}(x)$ and suppose $(X_i)_{i=1}^n$ is some orthonormal frame of M at x. Since $\mathrm{Iso}(M_{n;c}, \hat{g}_{n;c}) = G_c(n)$ acts transitively on the space of orthonormal frames of $\hat{M}_{n;c}$, there is an $\hat{F} \in \mathrm{Iso}(M_{n;c}, \hat{g}_{n;c})$ such that $\hat{F}_*(AX_i) = A'X_i$ for all $i = 1, \dots, n$. This implies that $\hat{F}(\hat{x}) = \hat{x}'$ and $\hat{F}_* A = A'$.

If $c = 0$ we set $B_{\hat{F}} = (\hat{x}' - \hat{F}_*|_{\hat{x}}(\hat{x}), \hat{F}_*|_{\hat{x}})$, where $\hat{F}_*|_{\hat{x}}$ is thought as a map $\mathbb{R}^n \to \mathbb{R}^n$ through canonical identifications of $T|_{\hat{x}} \hat{M}_{n;c}$ and $T|_{\hat{x}'} \hat{M}_{n;c}$ with $\mathbb{R}^n$. If $c \neq 0$ the element $B_{\hat{F}}$ of $G_c(n)$ is uniquely determined by setting it to be equal to $\hat{F}_*|_{\hat{x}}$ on $T|_{\hat{x}} \hat{M}_{n;c}$ and imposing that $B_{\hat{F}}(\hat{x}) = \hat{x}'$. Therefore, we get $\mu(B_{\hat{F}}, q) = q'$ which therefore shows the transitivity.

We first prove that if $\hat{F} \in \mathrm{Iso}(\hat{M}_{n;c}, \hat{g}_{n;c})$ and $B_{\hat{F}} \in G_c(n)$ as defined above, then $\mu(B_{\hat{F}}, q) = \hat{F} \cdot q$ where $q = (x, \hat{x}; A)$ and the right hand side is defined in Proposition 3.10. If $c = 0$, then

$$\begin{aligned}\mu(B_{\hat{F}}, q) =& \mu((\hat{F}(\hat{x}) - \hat{F}_*|_{\hat{x}}(\hat{x}), \hat{F}_*|_{\hat{x}}), q) = (x, \hat{F}_*|_{\hat{x}}(\hat{x}) + (\hat{F}(\hat{x}) - \hat{F}_*|_{\hat{x}}(\hat{x})); \hat{F}_*|_{\hat{x}} \circ A)) \\ =& (x, \hat{F}(\hat{x}); \hat{F}_*|_{\hat{x}} \circ A) = \hat{F} \cdot q,\end{aligned}$$

while if $c \neq 0$, $\mu(B_{\hat{F}}, q) = (x, B_{\hat{F}}(\hat{x}); B_{\hat{F}} \circ A) = (x, \hat{F}(\hat{x}); \hat{F}_*|_{\hat{x}} \circ A) = \hat{F} \cdot q$.

To prove the properness, consider a sequence B_n in $G_c(n)$ and $q_n = (x_n, \hat{x}_n; A_n)$ in Q such that $q_n \to q = (x, \hat{x}; A)$ and $\mu(B_n, q_n) \to q' = (x', \hat{x}'; A')$ as $n \to \infty$. Choose the unique $\hat{F}_n \in \mathrm{Iso}(M_{n;c}, \hat{g}_{n;c})$ such that $B_n = B_{\hat{F}_n}$ as above. Then $\mu(B_n, q_n) = \hat{F}_n \cdot q \to q'$ implies in particular that $\hat{F}_n(\hat{x}_n) \to \hat{x}'$ and we also have $\hat{x}_n \to \hat{x}$. Since the action of the isometry group of a complete connected Riemannian manifold is proper, we hence obtain a subsequence $\hat{F}_{n_i}$ of $\hat{F}_n$ converging to $\hat{F} \in \mathrm{Iso}(\hat{M}_{n;c}, \hat{g}_{n;c})$. Then B_{n_i} converges to $B_{\hat{F}}$ and we are done.

It remains to check the claim that the action μ preserves $\mathcal{D}_{\mathrm{R}}$ in the sense stated above. Let $B \in G_c(n)$. Since $\mathrm{Iso}(\hat{M}_{n;c}, \hat{g}_{n;c}) = G_c(n)$, there is a unique $\hat{F} \in \mathrm{Iso}(\hat{M}_{n;c}, \hat{g}_{n;c})$ such that $B = B_{\hat{F}}$ as above. Let $q = (x, \hat{x}; A) \in Q$ and let γ be any smooth curve in M such that $\gamma(0) = x$. By what was proved above and Proposition 3.10 imply that for all t,

$$\mu(B, q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q)(t)) = \hat{F} \cdot q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q)(t) = q_{\mathcal{D}_{\mathrm{R}}}(\gamma, \hat{F} \cdot q)(t) = q_{\mathcal{D}_{\mathrm{R}}}(\gamma, \mu(B, q))(t).$$

Taking derivative with respect to t at $t=0$, we find that

$$\begin{aligned}(\mu_B)_*\mathscr{L}_{\mathrm{R}}(\dot\gamma(0))|_q =&(\mu_B)_*\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 q_{\mathcal{D}_{\mathrm{R}}}(\gamma,q)(t)=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\mu(B,q_{\mathcal{D}_{\mathrm{R}}}(\gamma,q)(t))\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 q_{\mathcal{D}_{\mathrm{R}}}(\gamma,\mu(B,q))(t)=\mathscr{L}_{\mathrm{R}}(\dot\gamma(0))|_{\mu(B,q)}.\end{aligned}$$

This implies that $(\mu_B)_*\mathcal{D}_{\mathrm{R}}=\mathcal{D}_{\mathrm{R}}|_{\mu(B,q)}$ and hence allows us to conclude the proof of (i).

(ii) This follows from the general theory of principal bundle connections. See [13, 16]. □

4.2 Rolling Against an Euclidean Space

In this section, we give a necessary and sufficient condition for the controllability of $(\Sigma)_R$ in the case that $\hat M=\mathbb{R}^n$ equipped with the Euclidean metric i.e. $(\hat M,\hat g)=(\hat M_{n;0},\hat g_{n;0})$.

Now fix a point q_0 of $Q=Q(M,\mathbb{R}^n)$ of the form $q_0=(x_0,0;A_0)$ i.e., the initial contact point on M is equal to x_0 and, on $\mathbb{R}^n$, it is the origin. Since $\mathbb{R}^n$ is flat, for any a.c. curve $t\mapsto\hat\gamma(t)$ in $\mathbb{R}^n$ and $\hat X\in\mathbb{R}^n$ we have $P_0^t(\hat\gamma(t))\hat X=\hat X$, where we understand the canonical isomorphisms $T|_{\hat\gamma(0)}\mathbb{R}^n\cong\mathbb{R}^n\cong T|_{\hat\gamma(t)}\mathbb{R}^n$. We can then parameterize the rolling curves explicitly in the form:

$$q_{\mathcal{D}_{\mathrm{R}}}(\gamma,(x_0,\hat x;A))(t)=\Big(\gamma(t),\hat x+A\int_0^t P_s^0(\gamma)\dot\gamma(s)\mathrm{d}s;AP_t^0(\gamma)\Big),\tag{10}$$

for any $(x_0,0;A_0),(x_0,\hat x;A)\in Q$ and $\gamma\in\Omega_{x_0}(M)$.

We will make some standard observations for subgroups G of an Euclidean group $\mathrm{SE}(V)$, where (V,h) is a finite dimensional inner product space. Call an element of G of the form (v,id_V) a *pure translation* of G and write $T=T(G)$ for the set that they form. Clearly T is a subgroup of G. Let pr_1, pr_2 denote the projections $\mathrm{SE}(V)\to V$ and $\mathrm{SE}(V)\to\mathrm{SO}(V)$.

Proposition 4.2 Let G be a Lie subgroup of $\mathrm{SE}(V)$ with $\mathrm{pr}_2(G)=\mathrm{SO}(V)$. Then either of the following cases hold:

(i) $G=\mathrm{SE}(V)$ or

(ii) there exists $v^*\in V$ which is a fixed point of G.

Proof. Suppose first that $T=T(G)$ is non-trivial i.e., there exists a pure translation $(v,\mathrm{id}_V)\in T$, $v\neq0$. Then for any $(w,A)\in G$ it holds that

$$\begin{aligned}G\ni&(w,A)^{-1}\star(v,\mathrm{id}_V)\star(w,A)=(-A^{-1}w,A^{-1})\star(v+w,A)\\ &=(A^{-1}(v+w)-A^{-1}w,\mathrm{id}_V)=(A^{-1}v,\mathrm{id}_V),\end{aligned}$$

which implies that

$$\begin{aligned}T\supset\{(A^{-1}v,\mathrm{id}_V)\mid(w,A)\in G\}=&\{(A^{-1}v,\mathrm{id}_V)\mid A\in\mathrm{pr}_2(G)=\mathrm{SO}(V)\}\\ =&S^{n-1}(0,\|v\|)\times\{\mathrm{id}_V\}\end{aligned}$$

where $S^{n-1}(w,r)$, $w\in\mathbb{R}^n,r>0$ is the sphere of radius r centered at $w\in V$ and $\|\cdot\|=h(\cdot,\cdot)^{1/2}$. If $w\in V$ such that $\|w\|\leq\|v\|$ then it is clear that there are $u,u'\in S^{n-1}(0,\|v\|)$ such that $u+u'=w$ (choose $u\in S^{n-1}(0,\|v\|)\cap S^{n-1}(w,\|v\|)$ and $u'=w-u$). Therefore

$$(w,\mathrm{id}_V)=(u,\mathrm{id}_V)\star(u',\mathrm{id}_V)\in T$$

i.e., $\overline{B}(0,\|v\|)\subset T$ where $\overline{B}(w,r)$ is the closed ball of radius r centered at w. For $k\in\mathbb{N}$,

$$\begin{aligned}&\{\underbrace{\overline{B}(0,\|v\|)+\cdots+\overline{B}(0,\|v\|)}_{k\ \text{times}}\}\times\{\mathrm{id}_V\}\\ =&\underbrace{(\overline{B}(0,\|v\|)\times\{\mathrm{id}_V\})\star\cdots\star(\overline{B}(0,\|v\|)\times\{\mathrm{id}_V\})}_{k\ \text{times}}\subset T.\end{aligned}$$

From this we conclude that $V\times\{\mathrm{id}_V\}=T$. Therefore we get the case (i) since

$$\begin{aligned}G=&T\star G=\{(u,\mathrm{id}_V)\star(w,A)\mid u\in V,\ (w,A)\in G\}\\ =&\{(u+w,A)\mid u\in V,\ (w,A)\in G\}=\{(u,A)\mid u\in V,\ A\in\mathrm{pr}_2(G)=\mathrm{SO}(V)\}\\ =&V\times\mathrm{SO}(V)=\mathrm{SE}(V).\end{aligned}$$

The case that is left to investigate is the one where T is trivial i.e., $T=\{(0,\mathrm{id}_V)\}$. In this case the smooth surjective Lie group homomorphism $\mathrm{pr}_2|_G:G\to\mathrm{SO}(V)$ is also injective. In fact, if $A=\mathrm{pr}_2(v,A)=\mathrm{pr}_2(w,A)$ for $(v,A),(w,A)\in G$ and $v\neq w$, then

$$G\ni(w,A)\star(v,A)^{-1}=(w,A)\star(-A^{-1}v,A^{-1})=(w-v,\mathrm{id}_V)\in T,$$

and since $(w-v,\mathrm{id}_V)\neq(0,\mathrm{id}_V)$, this contradicts the triviality of T. It follows that $\mathrm{pr}_2|_G$ is a Lie group isomorphism onto $\mathrm{SO}(V)$ and hence a diffeomorphism. In particular, G is compact since $\mathrm{SO}(V)$ is compact.

Take a nonzero $v\in V$ and writing μ_{H} for the (right- and) left-invariant normalized (to 1) Haar measure of the compact group G, we define $v^*:=\int_G(B\star v)\mathrm{d}\mu_{\mathrm{H}}(B)$. Thus for $(w,A)\in G$,

$$\begin{aligned}(w,A)\star v^*=&w+Av^*=\int_G\big(w+A(B\star v)\big)\mathrm{d}\mu_{\mathrm{H}}(B)=\int_G\big(((w,A)\star B)\star v\big)\mathrm{d}\mu_{\mathrm{H}}(B)\\ =&\int_G(B\star v)\mathrm{d}\mu_{\mathrm{H}}(B)=v^*,\end{aligned}$$

where, in the second equality, we have used the linearity of the integral and normality of the Haar measure and in the last phase the left invariance of the Haar measure. This proves that v^* is a fixed point of G and completes the proof.

□

The previous proposition allows us prove the main theorem of this section.

Theorem 4.3 Suppose (M,g) is a complete Riemannian n-manifold and $(\hat{M},\hat{g})=\mathbb{R}^n$ is the Euclidean n-space. Then the rolling problem (R) is completely controllable if and only if the holonomy group of (M,g) is $\mathrm{SO}(n)$.

Proof. We write $H|_x$ for the holonomy group $H^\nabla|_x$ of ∇ at $x\in M$.

Suppose first that (R) is completely controllable. We need to show that $H|_{x_0}=\mathrm{SO}(T|_{x_0}M)$ for some given $x_0\in M$. Let $A_0:=\mathrm{id}_{T|_{x_0}M}$ and $q_0:=(x_0,0;A_0)\in Q$ where we understand the canonical identification $T|_0(T|_{x_0}M)=T|_{x_0}M$.

Given $B\in\mathrm{SO}(T|_{x_0}M)$, set $q=(x_0,0;AB)\in Q$. By assumption $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)=Q$ so there exists a $\gamma\in\Omega_{x_0}(M)$ (notice that $\pi_{Q,M}(q_0)=x_0=\pi_{Q,M}(q)$) such that $q'=q_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0)(1)$ which by (10) means that

$$(x_0,0;AB)=\Big(x_0,A\int_0^1P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s;AP_1^0(\gamma)\Big)$$

and thus $B=P_1^0(\gamma)\in H|_{x_0}$. This proves the necessity of the condition.

Assume now that the holonomy group of M is $\mathrm{SO}(n)$ i.e., for any $x \in M$ we have $H|_x = \mathrm{SO}(T|_x M)$. Let $q = (x, 0; A) \in Q$ and let $\mathcal{H}_q$ be the subgroup of $\mathrm{SE}(n)$ such that $\mu(\mathcal{H}_q \times \{q\}) = \pi_{Q,M}^{-1}(x) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ as in Proposition 4.1 case (ii).

We claim that $\mathrm{pr}_2(\mathcal{H}_q) = \mathrm{SO}(n)$. Indeed, if $B \in \mathrm{SO}(n)$, then $A^{-1}BA \in \mathrm{SO}(T|_x M) = H|_x$ and hence there is a $\gamma \in \Omega_x(M)$ such that $A^{-1}BA = P_1^0(\gamma)$. Let $(\hat{y}, C) \in \mathcal{H}_q$ by such that $\mu((\hat{y}, C), q) = q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q)(1)$. Then from (10) and the definition of μ in Proposition 4.1 we obtain

$$(\hat{y}, CA) = \big(A \int_0^1 P_0^s(\gamma)\dot{\gamma}(s)\mathrm{d}s, AP_1^0(\gamma)\big)$$

and hence $B = AP_0^1(\gamma)A^{-1} = C \in \mathrm{pr}_2(\mathcal{H}_q)$, which establishes the claim.

It follows from Proposition 4.2 that either (i) $\mathcal{H}_q = \mathrm{SE}(n)$ or (ii) there exists a fixed point $w_q^* \in \mathbb{R}^n$ of $\mathcal{H}_q$. If (i) holds for some $q_0 = (x_0, 0; A_0) \in Q$, then by Proposition 4.1 we obtain

$$\pi_{Q,M}^{-1}(x_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = \mu(\mathcal{H}_q \times \{q\}) = \mu(\mathrm{SE}(n) \times \{q\}) = \pi_{Q,M}^{-1}(x_0)$$

and hence $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = Q$ because $\pi_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0),M}$ is a subbundle of $\pi_{Q,M}$. Thus the rolling problem (R) is completely controllable if (i) holds.

Therefore suppose that (ii) holds i.e., for every $q \in Q$ of the form $q = (x, 0; A)$ there is a fixed point $w_q^* \in \mathbb{R}^n$ of $\mathcal{H}_q$. We will prove that this implies that (M, g) is flat which is a contradiction since (M, g) does not have a trivial holonomy group.

Thus for any point of Q of the form $q = (x, 0; A)$ and all loops $\gamma \in \Omega_x(M)$ we have by (10) and Proposition 4.1,

$$\begin{aligned}
w_q^* =&(\mu^q)^{-1}(q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q)(1)) \star w_q^* = (\mu^q)^{-1}\Big(x, \int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s; AP_1^0(\gamma)\Big) \star w_q^* \\
=&\Big(A\int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s, AP_1^0(\gamma)A^{-1}\Big) \star w_q^* = AP_1^0(\gamma)A^{-1}w_q^* + A\int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s.
\end{aligned}$$

In other words we have $(P_1^0(\gamma) - \mathrm{id})A^{-1}w_q^* + \int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s = 0$. Thus if $q = (x, 0; A)$ and $q' = (x, 0; A')$ are on the same π_Q-fiber over $(x, 0)$, then $(P_1^0(\gamma) - \mathrm{id})(A^{-1}w_q^* - A'^{-1}w_{q'}^*) = 0$ for every $\gamma \in \Omega_x(M)$. On the other hand, since M has full holonomy i.e., $H|_x = \mathrm{SO}(T|_x M)$, and $H|_x = \{P_1^0(\gamma) \mid \gamma \in \Omega_x(M)\}$, it follows from the above equation that $A^{-1}w_q^* = A'^{-1}w_{q'}^*$. This means that for every $x \in M$ there is a unique vector $V|_x \in T|_x M$ such that $V|_x = A^{-1}w_q^*, \quad \forall q \in \pi_Q^{-1}(x, 0)$. Moreover, the map $V : M \to TM;\ x \mapsto V|_x$ is a vector field on M (smoothness of V is deduced below) satisfying

$$P_1^0(\gamma)V|_x - V|_x = -\int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s, \quad \forall \gamma \in \Omega_x(M). \tag{11}$$

It follows from this that, for any piecewise C^1 path $\gamma : [0, 1] \to M$, we have

$$V|_{\gamma(1)} = P_0^1(\gamma)\Big(V|_{\gamma(0)} - \int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s\Big). \tag{12}$$

Indeed, if $\omega \in \Omega_{\gamma(1)}(M)$, then $\gamma^{-1}.\omega.\gamma \in \Omega_{\gamma(0)}(M)$ and therefore

$$\begin{aligned}
&P_1^0(\gamma)P_1^0(\omega)P_0^1(\gamma)V|_{\gamma(0)} - V|_{\gamma(0)} = P_1^0(\gamma^{-1}.\omega.\gamma)V|_{\gamma(0)} - V|_{\gamma(0)} \\
=& -\int_0^1 P_s^0(\gamma^{-1}.\omega.\gamma)\frac{\mathrm{d}}{\mathrm{d}s}(\gamma^{-1}.\omega.\gamma)(s)\mathrm{d}s \\
=& -\int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s - P_1^0(\gamma)\int_0^1 P_s^0(\omega)\dot{\omega}(s)\mathrm{d}s - P_1^0(\gamma)P_1^0(\omega)\int_0^1 P_s^0(\gamma^{-1})\frac{\mathrm{d}}{\mathrm{d}s}\gamma^{-1}(s)\mathrm{d}s \\
=& -\int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s + P_1^0(\gamma)(P_1^0(\omega)V|_{\gamma(1)} - V|_{\gamma(1)}) + P_1^0(\gamma)P_1^0(\omega)P_0^1(\gamma)\int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s,
\end{aligned}$$

that is $\left(P_1^0(\omega)-\mathrm{id}\right)P_0^1(\gamma)\Big(V|_{\gamma(0)}-\int_0^1 P_s^0(\gamma)\dot\gamma(s)\mathrm{d}s\Big)=\left(P_1^0(\omega)-\mathrm{id}\right)V|_{\gamma(1)}$. Equation (12) then follows from this since $\{P_1^0(\omega)\mid\omega\in\Omega_{\gamma(1)}(M)\}=H|_{\gamma(1)}=\mathrm{SO}(T|_{\gamma(1)}M)$.

Since (M,g) is complete, the geodesic $\gamma_X(t)=\exp_x(tX)$ is defined for all $t\in[0,1]$. Inserting this to Eq. (12) and noticing that $P_s^0(\gamma_X)\dot\gamma_X(s)=X$ in this case for all $s\in[0,1]$, we get $V|_{\gamma_X(1)}=P_0^1(\gamma_X)(V|_x-X)$. In particular, one deduces from this formula that V is smooth on M. If $X:=V|_x$ and $z:=\gamma_X(1)=\exp_x(V|_x)$, we get $V|_z=0$.

Fix $z\in M$ such that $V|_z=0$ and fix also some $q^*\in Q$ of the form $q^*=(z,0;A_0)$ (one may e.g. take $A_0=\mathrm{id}_{T|_zM}$). Equation (12) is clearly equivalent to

$$P_t^0(\gamma)V|_{\gamma(t)}=V|_{\gamma(0)}-\int_0^t P_s^0(\gamma)\dot\gamma(s)\mathrm{d}s$$

for any piecewise smooth path $\gamma:[0,T]\to M$, $T>0$. Taking γ to be smooth and differentiating the above equation w.r.t to t, we get $P_t^0(\gamma)\nabla_{\dot\gamma(t)}V=-P_t^0(\gamma)\dot\gamma(t)$, i.e., $\nabla_{\dot\gamma(t)}V=-\dot\gamma(t)$. Since γ was an arbitrary smooth curve, this implies that

$$\nabla_X V=-X,\quad \forall X\in\mathrm{VF}(M). \tag{13}$$

For any $X\in\mathrm{VF}(M)$, the vector $R(X,V)V$ can be seen to vanish everywhere since

$$\begin{aligned}R(X,V)V=&\nabla_X\nabla_V V-\nabla_V\nabla_X V-\nabla_{[X,V]}V=-\nabla_X V+\nabla_V X+[X,V]\\ =&[V,X]+[X,V]=0,\end{aligned}$$

where, in the second equality, we used (13).

For any $X\in T|_zM$, we write $\gamma_X(t)=\exp_z(tX)$ for the geodesic through z in the direction of X. It follows that

$$V|_{\gamma_X(t)}=P_0^t(\gamma_X)(V|_z-\int_0^t P_s^0(\gamma_X)\dot\gamma_X(s)\mathrm{d}s)=P_0^t(\gamma_X)(-\int_0^t X\mathrm{d}s)=P_0^t(\gamma_X)(-tX)=-t\dot\gamma_X(t).$$

Now for given $X,v\in T|_zM$ let $Y(t)=\frac{\partial}{\partial s}\big|_0\exp_z(t(X+sv))$ be the Jacobi field along γ_X such that $Y(0)=0$, $\nabla_{\dot\gamma_X(t)}Y|_{t=0}=v$. Then one has

$$\nabla_{\dot\gamma_X(t)}\nabla_{\dot\gamma_X}Y=R(\dot\gamma_X(t),Y(t))\dot\gamma_X(t)=\frac{1}{t^2}R(V|_{\gamma_X(t)},Y(t))V|_{\gamma_X(t)}=0,$$

for $t\neq 0$ which means that $t\mapsto\nabla_{\dot\gamma_X(t)}Y$ is parallel along γ_X i.e., $\nabla_{\dot\gamma_X(t)}Y=P_0^t(\gamma_X)\nabla_{\dot\gamma_X(0)}Y=P_0^t(\gamma_X)v$. This allows us to compute, for any t,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\|Y(t)\right\|_g^2=&2\frac{\mathrm{d}}{\mathrm{d}t}g(\nabla_{\dot\gamma_X(t)}Y,Y(t))=2g(\underbrace{\nabla_{\dot\gamma_X(t)}\nabla_{\dot\gamma_X}Y}_{=0},Y(t))+2g(\nabla_{\dot\gamma_X(t)}Y,\nabla_{\dot\gamma_X(t)}Y)\\ =&2g(P_0^t(\gamma_X)v,P_0^t(\gamma_X)v)=2\left\|v\right\|_g^2\end{aligned}$$

and then $\frac{\mathrm{d}}{\mathrm{d}t}\left\|Y(t)\right\|_g^2=2\left\|v\right\|_g^2t+\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\left\|Y(t)\right\|_g^2=2\left\|v\right\|_g^2t$, since $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\left\|Y(t)\right\|_g^2=2g(\nabla_{\dot\gamma_X(0)}Y,Y(0))=0$ as $Y(0)=0$. Therefore,

$$\left\|Y(t)\right\|_g^2=\left\|v\right\|_g^2t^2+\left\|Y(0)\right\|_g^2=\left\|v\right\|_g^2t^2,$$

which means that $\left\|t(\exp_z)_*|_{tX}(v)\right\|_g=\left\|tv\right\|_g$ and hence, when $t=1$,

$$\left\|(\exp_z)_*|_X(v)\right\|_g=\left\|v\right\|_g,\quad\forall X,v\in T|_zM. \tag{14}$$

This proves that $\exp_z$ is a local isometry $(T|_zM,g|_z)\to(M,g)$ and hence a Riemannian covering. Thus (M,g) is flat and the proof is finished.

□

Remark 4.4 For results and proofs in similar lines to those of the above Proposition and Theorem, see Theorem IV.7.1, p. 193 and Theorem IV.7.2, p. 194 in [16].

4.3 Rolling Against a Non-Flat Space Form

4.3.1 The Rolling Connection

Let $\pi_{TM\oplus\mathbb{R}} : TM \oplus \mathbb{R} \to M$ be the vector bundle over M where $\pi_{TM\oplus\mathbb{R}}(X,r) = \pi_{TM}(X)$. In this section we will prove the following result.

Theorem 4.5 There exists a vector bundle connection ∇^{Rol} of the vector bundle $\pi_{TM\oplus\mathbb{R}}$ that we call the *rolling connection*, and which we define as follows: for every $x \in M$, $Y \in T|_x M$, $X \in \mathrm{VF}(M)$, $r \in C^\infty(M)$,

$$\nabla^{\mathsf{Rol}}_Y(X,r) = \Big(\nabla_Y X + r(x)Y, Y(r) - cg\big(X|_x, Y\big)\Big), \tag{15}$$

such that in the case of (M,g) rolling against the space form $(\hat{M}_{n;c}, \hat{g}_{n;c})$, $c \neq 0$, the holonomy group G of $\mathcal{D}_{\mathrm{R}}$ is isomorphic to the holonomy group $H^{\nabla^{\mathsf{Rol}}}$ of ∇^{Rol}.

Moreover, if one defines a fiber inner product h_c on $TM \oplus \mathbb{R}$ by

$$h_c((X,r),(Y,s)) = g(X,Y) + c^{-1}rs,$$

where $X, Y \in T|_x M$, $r, s \in \mathbb{R}$, then ∇^{Rol} is a metric connection in the sense that for every $X, Y, Z \in \mathrm{VF}(M)$, $r, s \in C^\infty(M)$,

$$Z\big(h_c((X,r),(Y,s))\big) = h_c(\nabla^{\mathsf{Rol}}_Z(X,r),(Y,s)) + h_c((X,r),\nabla^{\mathsf{Rol}}_Z(Y,s)).$$

Before providing the proof of the theorem, we present the equations of parallel transport w.r.t ∇^{Rol} along a general curve and along a geodesic of M and also the curvature of ∇^{Rol}. Let $\gamma : [0,1] \to M$ be an a.c. curve on M, $\gamma(0) = x$ and let $(X_0, r_0) \in T|_x M \oplus \mathbb{R}$. Then the parallel transport $(X(t), r(t)) = (P^{\nabla^{\mathsf{Rol}}})^t_0(\gamma)(X_0, r_0)$ of (X_0, r_0) is determined from the equations

$$\begin{cases} \nabla_{\dot\gamma(t)} X + r(t)\dot\gamma(t) = 0, \\ \dot r(t) - cg(\dot\gamma(t), X(t)) = 0, \end{cases} \tag{16}$$

for a.e. $t \in [0,1]$. In particular, if γ is a geodesic on (M,g), one may derive the following uncoupled second order differential equations for X and r, for all t,

$$\begin{cases} \nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)} X + cg(X(t), \dot\gamma(t))\dot\gamma(t) = 0, \\ \ddot r(t) + c\,\|\dot\gamma(t)\|_g^2\, r(t) = 0. \end{cases} \tag{17}$$

One easily checks that the connection ∇^{Rol} on $\pi_{TM\oplus\mathbb{R}}$ has the curvature,

$$R^{\nabla^{\mathsf{Rol}}}(X,Y)(Z,r) = \big(R(X,Y)Z - c(g(Y,Z)X - g(X,Z)Y), 0\big), \tag{18}$$

where $X, Y, Z \in \mathrm{VF}(M)$, $r \in C^\infty(M)$.

Proof. We have proved in Proposition4.1 that the rolling distribution $\mathcal{D}_{\mathrm{R}}$ is a principal bundle connection for the principal $G_c(n)$-bundle $\pi_{Q,M} : Q \to M$. By a standard procedure (cf. Definition 2.1.3 and Proposition 2.3.7 in [13]), the previous fact implies that there is a vector bundle $\xi : E \to M$ with fibers isomorphic to $\mathbb{R}^{n+1}$ and a unique linear vector bundle connection $\nabla^{\mathsf{Rol}} : \Gamma(\xi) \times \mathrm{VF}(M) \to \Gamma(\xi)$ which induces the distribution $\mathcal{D}_{\mathrm{R}}$ on Q. Then the holonomy group G of $\mathcal{D}_{\mathrm{R}}$ and $H^{\nabla^{\mathsf{Rol}}}$ of ∇^{Rol} are isomorphic. We will eventually show that ξ is further isomorphic to $\pi_{TM\oplus\mathbb{R}}$ and give the explicit expression (15) for the connection of $\pi_{TM\oplus\mathbb{R}}$ induced by this isomorphims from ∇^{Rol} on ξ.

There is a canonical non-degenerate metric $h_c : E \odot E \to M$ on the vector bundle ξ (positive definite when $c > 0$ and indefinite if $c < 0$) and the connection ∇^{Rol} is a metric connection w.r.t. to h_c i.e., for any $Y \in \mathrm{VF}(M)$ and $s, \sigma \in \Gamma(\nu)$,

$$Y\big(h_c(s,\sigma)\big) = h_c(\nabla^{\mathsf{Rol}}_Y s, \sigma) + h_c(s, \nabla^{\mathsf{Rol}}_Y \sigma). \tag{19}$$

The construction of ξ goes as follows (see [13], section 2.1.3). Define a left $G_c(n)$-group action β on $Q \times \mathbb{R}^{n+1}$ by $\beta(B,(q,v)) = (\mu(B,q), Bv)$, where $q \in Q$, $v \in \mathbb{R}^{n+1}$, $B \in G_c(n)$. The action β is clearly smooth, free and proper. Hence $E := (Q \times \mathbb{R}^{n+1})/\beta$ is a smooth manifold of dimension $n + (n+1) = 2n+1$. The β-equivalence classe (i.e., β-orbit) of $(q,v) \in Q \times \mathbb{R}^{n+1}$ is denoted by $[(q,v)]$. Then one defines $\xi\big([(q,v)]\big) = \pi_{Q,M}(q)$ which is well defined since the β-action preserves the fibers of $Q \times \mathbb{R}^{n+1} \to M$; $(q,v) \mapsto \pi_{Q,M}(q)$. We prove now that ξ is isomorphic, as a vector bundle over M, to

$$\begin{aligned} \pi_{TM\oplus\mathbb{R}} : TM \oplus \mathbb{R} &\to M, \\ (X,t) &\mapsto \pi_{TM}(X). \end{aligned}$$

Indeed, let $f \in \Gamma(\xi)$ and notice that for any $q \in Q$ there exists a unique $\overline{f}(q) \in \mathbb{R}^{n+1}$ such that $[(q, \overline{f}(q))] = f(\pi_{Q,M}(q)))$ by the definition of the action β. Then $\overline{f} : Q \to \mathbb{R}^{n+1}$ is well defined and, for each $q = (x,\hat{x};A)$, there are unique $X|_q \in T|_x M$, $r(q) \in \mathbb{R}$ such that

$$\overline{f}(q) = AX|_q + r(q)\hat{x}.$$

The maps $q \mapsto X|_q$ and $q \mapsto r(q)$ are smooth. We show that the vector $X|_q$ and the real number $r(q)$ depend only on x and hence define a vector field and a function on M. One has $[((x,\hat{x};A),v)] = [((x,\hat{y};B),w)]$ if and only if there is $C \in G_c(n)$ such that $C\hat{x} = \hat{y}$, $CA = B$ and $Cv = w$. This means that $C|_{\mathrm{im}A} = BA^{-1}|_{\mathrm{im}A} : T|_{\hat{x}}\hat{M}_{n;c} \to T|_{\hat{y}}\hat{M}_{n;c}$ (with $\mathrm{im}A$ denoting the image of A) and this defines C uniquely as an element of $G_c(n)$ and also, by the definition of $\overline{f}$, $C\overline{f}(x,\hat{x},A) = \overline{f}(x,\hat{y},B)$. Therefore,

$$BX|_{(x,\hat{y};B)} + r(x,\hat{y};B)\hat{y} = C(AX|_{(x,\hat{x};A)} + r(x,\hat{x};A)\hat{x}) = BX|_{(x,\hat{x};A)} + r(x,\hat{x};A)\hat{y},$$

which shows that $X|_{(x,\hat{y};B)} = X|_{(x,\hat{x};A)}$, $r(x,\hat{y};B) = r(x,\hat{x};A)$ and proves the claim.

Hence for each $f \in \Gamma(\xi)$ there are unique $X_f \in \mathrm{VF}(M)$ and $r_f \in C^\infty(M)$ such that

$$f(x) = \big[\big((x,\hat{x};A), AX_f|_x + r_f(x)\hat{x}\big)\big],$$

(here the right hand side does not depend on the choice of $(x,\hat{x};A) \in \pi_{Q,M}^{-1}(x)$).

Conversely, given $X \in \mathrm{VF}(M)$, $r \in C^\infty(M)$ we may define $f_{(X,r)} \in \Gamma(\xi)$ by

$$f_{(X,r)}(x) = \big[\big((x,\hat{x};A), AX|_x + r(x)\hat{x}\big)\big],$$

where the right hand side does not depend on the choice of $(x,\hat{x};A) \in \pi_{Q,M}^{-1}(x)$.

Clearly, for $f \in \Gamma(\xi)$, one has $f_{(X_f,r_f)} = f$ and, for $(X,r) \in \mathrm{VF}(M) \times C^\infty(M)$, one has $(X_{f_{(X,r)}}, r_{f_{(X,r)}}) = (X,r)$. This proves that the map defined by

$$\begin{aligned} \Gamma(\xi) &\to \mathrm{VF}(M) \times C^\infty(M) \\ f &\mapsto (X_f, r_f) \end{aligned}$$

is a bijection. It is easy to see that it is actually a $C^\infty(M)$-module homomorphism. Since $C^\infty(M)$-modules $\Gamma(\xi)$ and $\mathrm{VF}(M) \times C^\infty(M)$ are isomorphic and since $\mathrm{VF}(M) \times C^\infty(M)$ is obviously isomorphic, as a $C^\infty(M)$-module, to $\Gamma(\pi_{TM\oplus\mathbb{R}})$, it follows that ξ and $\pi_{TM\oplus\mathbb{R}}$ are isomorphic vector bundles over M.

We now describe the connection ∇^{Rol} and the inner product structure h_c on ξ and we determine to which objects they correspond to in the isomorphic bundle $\pi_{TM\oplus\mathbb{R}}$.

By Section 2.1.3 in [13] and the above notation, one defines for $f\in\Gamma(\xi)$, $Y\in T|_xM$, $x\in M$

$$\nabla^{\mathsf{Rol}}_Y f|_x := \big[\big((x,\hat{x};A),\mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}\overline{f}\big)\big],$$

where $\overline{f}:Q\to\mathbb{R}^{n+1}$ is defined above and $\mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}\overline{f}$ is defined componentwise (i.e., we let $\mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}$ to operate separately to each of the $n+1$ component functions of $\overline{f}$). The definition does not depend on $(x,\hat{x};A)\in\pi_{Q,M}^{-1}(x)$ as should be evident from the above discussions. The inner product on ξ, on the other hand, is defined by

$$h_c([((x,\hat{x};A),v)],[((x,\hat{y};B),w)]) = g(X,Y)+c^{-1}rt,$$

where $v=AX+r\hat{x}$, $w=BY+t\hat{y}$. It is clear that h_c is well defined.

We work out the expression for ∇^{Rol}. For clarity, we write $\iota:\hat{M}_{n;c}\to\mathbb{R}^{n+1}$ for the inclusion. Let $f\in\Gamma(\xi)$, $Y\in T|_xM$, $x\in M$. Then $\overline{f}(y,\hat{y},B)=\iota_*(BX_f|_y)+r_f(y)\hat{y}$ where $X_f\in\mathrm{VF}(M)$, $r_f\in C^\infty(M)$ and

$$\mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}\overline{f} = \mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}\big((y,\hat{y};B)\mapsto\iota_*(BX_f|_y)\big)+Y(r_f)\hat{x}+r_f(x)AY$$

Take a path γ on M such that $\dot{\gamma}(0)=Y$. Then $\dot{q}_{\mathcal{D}_{\mathrm{R}}}(\gamma,q)(0)=\mathscr{L}_{\mathrm{R}}(Y)|_q$, where $q=(x,\hat{x};A)$, and $\mathscr{L}_{\mathrm{R}}(Y)|_q\big((y,\hat{y};B)\mapsto\iota_*(BX_f|_y)\big)=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\iota_*\big(A_{\mathcal{D}_{\mathrm{R}}}(\gamma,q)(t)X_f|_{\gamma(t)}\big)$. Since

$$\begin{aligned}
&s_{n;c}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0\iota_*(A_{\mathcal{D}_{\mathrm{R}}}(\gamma,q)(t)X_f|_{\gamma(t)}),\hat{x}\Big)\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 s_{n;c}\big(\iota_*A_{\mathcal{D}_{\mathrm{R}}}(\gamma,q)(t)X_f|_{\gamma(t)},\hat{\gamma}_{\mathcal{D}_{\mathrm{R}}}(\gamma,q)(t)\big)-s_{n;c}\big(\iota_*AX_f|_x,\iota_*AY\big)\\
=&-\hat{g}_{n;c}(AX_f|_x,AY)=-g\big(X_f|_x,Y\big)=s_{n;c}(-cg\big(X_f|_x,Y\big)\hat{x},\hat{x}).
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathscr{L}_{\mathrm{R}}(Y)|_q\big((y,\hat{y};B)\mapsto\iota_*(BX_f|_y)\big)=&\iota_*\hat{\nabla}_{AY}(A_{\mathcal{D}_{\mathrm{R}}}(\gamma,q_0)(\cdot)X_f)-cg\big(X_f|_x,Y\big)\hat{x}\\
=&\iota_*A\nabla_YX_f-cg\big(X_f|_x,Y\big)\hat{x}
\end{aligned}$$

and so $\mathscr{L}_{\mathrm{R}}(Y)|_q\overline{f}=\iota_*A\big(\nabla_YX_f+r_f(x)Y\big)+\big(Y(r_f)-cg\big(X_f|_x,Y\big)\big)\hat{x}$.

Correspondingly, using the isomorphism of ξ and $\pi_{TM\oplus\mathbb{R}}$, to the connection ∇^{Rol} and the non-degenerate metric h_c on ξ, there is a connection ∇^{Rol} and an indefinite metric h_c (with the same names as the ones on ξ) on $\pi_{TM\oplus\mathbb{R}}$ such that for $X\in\mathrm{VF}(M)$, $r\in C^\infty(M)$ and $Y\in T|_xM$,

$$\nabla^{\mathsf{Rol}}_Y(X,r)=\Big(\nabla_YX+r(x)Y,Y(r)-cg\big(X|_x,Y\big)\Big), \tag{20}$$

where $(x,\hat{x};A)\in Q$ is any point of $\pi^{-1}x$ and $h_c((X,r),(Y,s))=g(X,Y)+c^{-1}rs$ for $X,Y\in T|_xM$, $r,s\in\mathbb{R}$. To finish the proof, we need to show that ∇^{Rol} is metric w.r.t. h_c. Indeed, if $X,Y,Z\in\mathrm{VF}(M)$, $r,s\in C^\infty(M)$, we get

$$\begin{aligned}
&h_c(\nabla^{\mathsf{Rol}}_Z(X,r),(Y,s))+h_c((X,r),\nabla^{\mathsf{Rol}}_Z(Y,s))\\
=&h_c\big((\nabla_ZX+rZ,Z(r)-cg(X,Z)),(Y,s)\big)+h_c\big((X,r),(\nabla_ZY+sZ,Z(s)-cg(Y,Z))\big)\\
=&g(\nabla_ZX+rZ,Y)+c^{-1}Z(r)s-g(X,Z)s+g(\nabla_ZY+sZ,X)+c^{-1}rZ(s)-rg(Y,Z)\\
=&g(\nabla_ZX,Y)+g(X,\nabla_ZY)+c^{-1}Z(r)s+c^{-1}rZ(s)=Z(g(X,Y)+c^{-1}rs)=Z\big(h_c((X,r),(Y,s))\big).
\end{aligned}$$

□

4.4 Rolling Holonomy for a Space Form of Positive Curvature

In this section, we assume that $c = 1$ i.e. $(\hat{M}, \hat{g}) = (\hat{M}_{n;1}, \hat{g}_{n;1})$ is the n-dimensional unit sphere S^n. It is now clear, thanks to Theorem 4.5, that the controllability of the rolling problem of a manifold M against the sphere S^n amounts to checking whether the connection ∇^{Rol} of $\pi_{TM\oplus\mathbb{R}}$ has full holonomy or not i.e., whether $H^{\mathsf{Rol}} := H^{\nabla^{\mathsf{Rol}}}$ is $\mathrm{SO}(n+1)$ or not.

The classical investigation of the holonomy group H of the Levi-Civita connection in Riemannian geometry is divided into several steps. The first one consists of studying the reducibility of the action of H^{Rol} and this issue is tackled by de Rham theorem (see [24]). The second step then deals with the question of transitivity of the irreducible action of H on the unit sphere. In particular, if this action is not transitive, the corresponding Riemannian manifold is shown to be (locally) symmetric (cf. [26], [13], [22], [4]). Then, from the list of compact connected subgroups of $\mathrm{SO}(n)$ having a transitive action on the unit sphere, one proceeds by either excluding candidates or constructing examples of manifolds having a prescribed holonomy group.

As regards to H^{Rol} the situation turns out to be much more simple and is summarized in the following theorem.

Theorem 4.6 Let ∇^{Rol} be the rolling connection associated to the rolling problem (R) of a complete simply connected Riemannian manifold (M, g) onto the unit sphere S^n, $n \geq 2$. We use H^{Rol} to denote the holonomy group of ∇^{Rol}. Assume that the action of H^{Rol} on the unit sphere is not transitive, then (M, g) admits the unit sphere as its universal covering space.

We deduce from the list of compact connected subgroups of $\mathrm{SO}(n)$ having a transitive action on the unit sphere (cf. [13], section 3.4.3) an immediate corollary regarding the complete controllability of the rolling problem (R) associated to the manifolds (M, g), $(\hat{M}, \hat{g})$ as in the previous theorem. Indeed, a transitive action on the unit sphere S^k for even dimension $k \geq 16$ corresponds to a unique compact connected subgroup of $\mathrm{SO}(k+1)$, namely $\mathrm{SO}(k+1)$ itself.

Corollary 4.7 If n is even and $n \geq 16$, then the rolling problem (R) associated to a complete simply connected (M, g) against a space form $(\hat{M}_{n;c}, \hat{g}_{n;c})$ of positive curvature $c > 0$ is completely controllable if and only if (M, g) is not of constant curvature c.

The proof of Theorem 4.6 is divided in two steps. We first assume that the action of H^{Rol} is reducible and then deal with the case of irreducible and non-transitive action.

4.5 Reducibility

Theorem 4.8 Let (M, g) be a complete connected Riemannian manifold and $(\hat{M}, \hat{g}) = S^n$ be the unit sphere. If the rolling holonomy group H^{Rol} corresponding to the rolling of (M, g) against S^n acts reducibly, then S^n is a Riemannian covering of (M, g) .

Proof. We write $h = h_1$ for the inner product on $TM \oplus \mathbb{R}$. Fix once and for all a point $x_0 \in M$. Since H^{Rol} acts reducibly, then there are two nontrivial subspaces $V_1, V_2 \subset T|_{x_0}M \oplus \mathbb{R}$ and invariant by the action of $H^{\mathsf{Rol}}|_{x_0}$, the holonomy group of ∇^{Rol} at x_0. Since the holonomy group of ∇^{Rol} acts h-orthogonally on $T|_{x_0}M$, it follows that $V_1 \perp V_2$.

Define subbundles $\pi_{\mathcal{D}_j} : \mathcal{D}_j \to M$, $j = 1, 2$ of $\pi_{TM\oplus\mathbb{R}}$ such that for any $x \in M$ one chooses a piecewise C^1 curve $\gamma : [0, 1] \to M$ from x_0 to x and defines

$$\mathcal{D}_j|_x = (P^{\nabla^{\mathsf{Rol}}})_0^1(\gamma)V_j, \quad j = 1, 2.$$

These definitions are independent of the chosen path γ: if ω is another curve, then $\omega^{-1}.\gamma \in \Omega_{x_0}(M)$ is a loop based at x_0 and by the invariance of V_j, $j=1,2$ under the holonomy of ∇^{Rol},

$$(P^{\nabla^{\mathsf{Rol}}})_0^1(\gamma)V_j = (P^{\nabla^{\mathsf{Rol}}})_0^1(\omega)\underbrace{(P^{\nabla^{\mathsf{Rol}}})_0^1(\omega^{-1}.\gamma)V_j}_{=V_j} = (P^{\nabla^{\mathsf{Rol}}})_0^1(\omega)V_j.$$

Moreover, since parallel transport $(P^{\nabla^{\mathsf{Rol}}})_0^1(\gamma)$ is an h-orthogonal map, it follows that $\mathcal{D}_1 \perp \mathcal{D}_2$ w.r.t the vector bundle metric h.

It is a standard fact that $\mathcal{D}_j$, $j=1,2$, are smooth embedded submanifolds of $TM\oplus\mathbb{R}$ and that the restriction of $\pi_{TM\oplus\mathbb{R}}$ to $\mathcal{D}_j$ defines a smooth subbundle $\pi_{\mathcal{D}_j}$ as claimed. Moreover, it is clear that $\pi_{\mathcal{D}_1}\oplus\pi_{\mathcal{D}_2}=\pi_{TM\oplus\mathbb{R}}$, and this sum is h-orthogonal.

We will now assume that both $\mathcal{D}_j$, $j=1,2$, have dimension at least 2. The case where one of them has dimension $=1$ can be treated in a similar fashion and will be omitted. So we let $m+1=\dim\mathcal{D}_1$ where $m\geq 1$ and then $n-m=(n+1)-(m+1)=\dim\mathcal{D}_2\geq 2$ i.e., $1\leq m\leq n-2$. Define for $j=1,2$

$$\mathcal{D}_j^M = \mathrm{pr}_1(\mathcal{D}_j) = \{X \mid (X,r)\in\mathcal{D}_j\}\subset TM,$$
$$N_j = \{x\in M \mid (0,1)\in\mathcal{D}_j|_x\}\subset M.$$

Trivially, $N_1\cap N_2=\emptyset$. Also, N_j, $j=1,2$, are closed subsets of M since they can be written as $N_j=\{x\in M \mid p_j^{\perp}(T|_x)=T|_x\}$ where $p_j^{\perp}: TM\oplus\mathbb{R}\to\mathcal{D}_j$ is the h-orthogonal projection onto $\mathcal{D}_j$ and T is the (smooth) constant section $x\mapsto(0,1)$ of $\pi_{TM\oplus\mathbb{R}}$.

We next provide a sketch of the proof. We show that N_j are nonempty totally geodesic submanifolds of M and, for any $x_j\in N_j$, $j=1,2$, that (M,g) is locally isometric to the sphere

$$S=\{(X_1,X_2)\in T|_{x_1}^{\perp}N_1\oplus T|_{x_2}^{\perp}N_2 \mid \|X_1\|_g^2+\|X_2\|_g^2=1\},$$

with the metric $G:=(g|_{T|_{x_1}^{\perp}N_1}\oplus g|_{T|_{x_2}^{\perp}N_2})|_S$. Here $\perp$ denotes the orthogonal complement inside $T|_xM$ w.r.t. g. Since (S,G) is isometric to the Euclidean sphere $(S^n,s_{n;1})$ this would finish the argument. The latter is rather long and we decompose it in a sequence of ten lemmas.

Lemma 4.9 The sets N_j, $j=1,2$, are non-empty.

Proof. Note that $N_1\cup N_2\neq M$ since otherwise $N_1=M\backslash N_2$ would be open and closed and similarly for N_2. If (say) $N_1\neq\emptyset$, then $N_1=M$ by connectedness of M i.e., the point $(0,1)\in\mathcal{D}_1|_x$ for all $x\in M$. Then, for all $x\in M$, $X\in\mathrm{VF}(M)$, $\mathcal{D}_1|_x\ni\nabla^{\mathsf{Rol}}_{X|_x}(0,1)=(X|_x,0)$, by the invariance of $\mathcal{D}_1$, the holonomy of ∇^{Rol} and (15), implying that $\mathcal{D}_1=TM\oplus\mathbb{R}$, a contradiction.

Let $x'\in M\backslash(N_1\cup N_2)$ be arbitrary. Choose a basis $(X_0,r_0),\dots(X_m,r_m)$ of $\mathcal{D}_1|_{x'}$. Then at least one of the numbers $r_0,\dots,r_m$ is non-zero, since otherwise one would have $(X_i,r_i)=(X_i,0)\perp(0,1)$ for all i and thus $\mathcal{D}_1|_{x'}\perp(0,1)$ i.e., $(0,1)\in\mathcal{D}_2|_{x'}$ i.e., $x'\in N_2$ which is absurd. We assume that it is r_0 which is non-zero. By taking appropriate linear combinations of (X_i,r_i), $i=0,\dots,m$ (and by Gram-Schmidt's process), one may change the basis (X_i,r_i), $i=0,\dots,m$, of $\mathcal{D}_1|_x$ so that $r_1,\dots,r_m=0$, $r_0\neq 0$ and that $(X_0,r_0),(X_1,0)\dots,(X_m,0)$ are h-orthonormal. Also, $X_0,\dots,X_m$ are non-zero: for $X_1,\dots,X_m$ this is evident, and for X_0 it follows from the fact that if $X_0=0$, then $r_0=1$ and hence $x'\in N_1$, which contradicts our choice of x'.

Now let $\gamma:\mathbb{R}\to M$ be the unit speed geodesic with $\gamma(0)=x'$, $\dot\gamma(0)=\frac{X_0}{\|X_0\|_g}$. Parallel translate (X_i,r_i) along γ by ∇^{Rol} to get $\pi_{\mathcal{D}_1}$-sections $(X_i(t),r_i(t))$ along γ. In particular, from (17) one gets $\ddot r_i(t)+r_i(t)=0$, with $r_0(0)\neq 0$, $r_1(0)=\dots=r_m(0)=0$. From the second equation in (16) one obtains $\dot r_i(0)=g(\dot\gamma(0),X_i(0))=\|X_0\|_g^{-1}g(X_0,X_i)$ and thus $\dot r_i(0)=0$ for $i=1,\dots,m$ since $(X_i,0)$ is h-orthogonal to (X_0,r_0). Moreover, $\dot r_0(0)=\|X_0\|_g$. Hence $r_i(t)=0$ for all t and $i=1,\dots,m$ and $r_0(t)=\|X_0\|_g\sin(t)+r_0\cos(t)$. In particular, at $t=$

$t_0 := \arctan(-\frac{r_0}{\|X_0\|_g})$ one has $r_i(t_0) = 0$ for all $i = 0, \dots, m$ which implies that $\mathcal{D}_1|_{\gamma(t_0)} \perp (0,1)$ i.e., $\gamma(t_0) \in N_2$. This proves that N_2 is non-empty. The same argument with $\mathcal{D}_1$ and $\mathcal{D}_2$ interchanged shows that N_1 is non-empty.

□

Lemma 4.10 For any $x \in M$ and any unit vector $u \in T|_x M$,

$$(P^{\nabla^{\mathsf{Rol}}})_0^t(\gamma_u)(0,1) = (-\sin(t)\dot{\gamma}_u(t), \cos(t)). \tag{21}$$

Proof. Here and in what follows, $\gamma_u(t) := \exp_x(tu)$. Write $(X_0(t), r_0(t)) := (P^{\nabla^{\mathsf{Rol}}})_0^t(\gamma_u)(0,1)$. The second equation in (16) implies that $\dot{r}_0(0) = g(\dot{\gamma}_u(0), X_0(0)) = g(u, 0) = 0$ and, since $r_0(0) = 1$, the second equation in (17) gives $r_0(t) = \cos(t)$. Notice that, for all $t \in \mathbb{R}$,

$$\begin{aligned}\nabla_{\dot{\gamma}_u(t)}(-\sin(t)\dot{\gamma}_u(t)) + r_0(t)\dot{\gamma}_u(t) &= \nabla_{\dot{\gamma}_u(t)}(-\sin(t))\dot{\gamma}_u(t) - \sin(t)\nabla_{\dot{\gamma}_u(t)}\dot{\gamma}_u(t) + \cos(t)\dot{\gamma}_u(t) \\ &= -\cos(t)\dot{\gamma}_u(t) - 0 + \cos(t)\dot{\gamma}_u(t) = 0,\end{aligned}$$

i.e., $-\sin(t)\dot{\gamma}_u(t)$ solves the same first order ODE as $X_0(t)$, $\nabla_{\dot{\gamma}_u(t)} X_0 + r_0(t)\dot{\gamma}_u(t) = 0$ by the first equation in (16). Moreover, since $\big(-\sin(t)\dot{\gamma}_u(t)\big)|_{t=0} = 0 = X_0(0)$, then $X_0(t) = -\sin(t)\dot{\gamma}_u(t)$, which, combined with the fact that $r_0(t) = \cos(t)$ proven above, gives (21).

□

Lemma 4.11 The sets N_j, $j = 1, 2$, are complete, totally geodesic submanifolds of (M, g) and $\mathcal{D}_j^M|_x = T|_x N_j$, $\forall x \in N_j$, $j = 1, 2$.

Proof. We show this for N_1. The same argument then proves the claim for N_2. Let $x \in N_1$ and $u \in \mathcal{D}_1^M|_x$ a unit vector. Since $(0,1) \in \mathcal{D}_1|_x$, Eq. (21) implies that

$$\mathcal{D}_1|_{\gamma_u(t)} \ni (P^{\nabla^{\mathsf{Rol}}})_0^t(\gamma_u)(0,1) = (-\sin(t)\dot{\gamma}_u(t), \cos(t)).$$

Next notice that

$$\begin{aligned}\nabla^{\mathsf{Rol}}_{\dot{\gamma}_u(t)}\big(\cos(t)\dot{\gamma}_u(t), \sin(t)\big) =& \big(-\sin(t)\dot{\gamma}_u(t) + \sin(t)\dot{\gamma}_u(t), \cos(t) - g(\dot{\gamma}_u(t), \cos(t)\dot{\gamma}_u(t))\big) \\ =& (0,0),\end{aligned}$$

and hence, since $\big(\cos(t)\dot{\gamma}_u(t), \sin(t)\big)|_{t=0} = (u, 0) \in \mathcal{D}_1|_x$ (because $u \in \mathcal{D}_1^M|_x$, hence there is some $r \in \mathbb{R}$ such that $(u, r) \in \mathcal{D}_1|_x$ and since $(0,1) \in \mathcal{D}_1|_x$, then $\mathcal{D}_1|_x \ni (u, r) - r(0,1) = (u, 0)$), we have, for all $t \in \mathbb{R}$, $\big(\cos(t)\dot{\gamma}_u(t), \sin(t)\big) = (P^{\nabla^{\mathsf{Rol}}})_0^t(u, 0) \in \mathcal{D}_1|_{\gamma_u(t)}$. and then

$$\mathcal{D}_1|_{\gamma_u(t)} \ni \sin(t)\big(\cos(t)\dot{\gamma}_u(t), \sin(t)\big) + \cos(t)\big(-\sin(t)\dot{\gamma}_u(t), \cos(t)\big) = (0,1).$$

This proves that any geodesic starting from a point of N_1 with the initial direction from $\mathcal{D}_1^M$ stays in N_1 forever. Hence, once it has been shown that N_1 is a submanifold of M with tangent space $T|_x N_1 = \mathcal{D}_1^M|_x$ for all $x \in N_1$, then automatically N_1 is totally geodesic and complete.

Let $x \in N_1$. If U is an open neighbourhood of x and (X_{m+1}, r_{m+1}), $\dots$, (X_n, r_n) local $\pi_{\mathcal{D}_2}$-sections forming a basis of $\mathcal{D}_2$ over U, then $N_1 \cap U = \{x \in U \mid r_{m+1}(x) = \dots = r_n(x) = 0\}$

Thus let $(X_{m+1}, r_{m+1}), \dots, (X_n, r_n) \in \mathcal{D}_2|_x$ be a basis of $\mathcal{D}_2|_x$. Choose $\epsilon > 0$ such that $\exp_x$ is a diffeomorphism from $B_g(0, \epsilon)$ onto its image U_ϵ and define for $y \in U_\epsilon$, $j = m+1, \dots, n$,

$$(X_j, r_j)|_y = (P^{\nabla^{\mathsf{Rol}}})_0^1\big(\tau \mapsto \exp_x(\tau \exp_x^{-1}(y))\big)(X_j, r_j).$$

Then (X_j, r_j) are local $\pi_{\mathcal{D}_2}$-sections and it is clear that

$$N_1 \cap U_\epsilon = \{y \in U_\epsilon \mid r_{m+1}(y) = \dots = r_n(y) = 0\}.$$

Moreover, from (16), $\nabla r_j|_x = X_j|_x, \quad j = m+1,\dots,n$, which are linearly independent. Hence, by taking $\epsilon > 0$ possibly smaller, we may assume that the local vector fields ∇r_j, $j = m+1,\dots,n$, are linearly independent on U_ϵ. But this means that $N_1 \cap U_\epsilon = \{y \in U_\epsilon \mid r_{m+1}(y) = \cdots = r_n(y) = 0\}$ is a smooth embedded submanifold of U_ϵ with tangent space

$$\begin{aligned}T|_x N_1 =&\{X \in T|_x M \mid g(\nabla r_j, X) = 0,\ j = m+1,\dots,n\}\\ =&\{X \in T|_x M \mid g(X_j, X) = 0,\ j = m+1,\dots,n\} = \mathcal{D}_1^M|_x.\end{aligned}$$

Since $x \in N_1$ was arbitrary, this proves that N_1 is indeed an embedded submanifold of M and $T|_x N_1 = \mathcal{D}_1^M|_x$ for all $x \in N_1$.

□

Lemma 4.12 Let $d_i(x) := d_g(N_i, x)$, $x \in M$. Then in the set where d_i is smooth,

$$(\nabla \cos(d_i(\cdot)), \cos(d_i(\cdot))) \in \mathcal{D}_i^M, \tag{22}$$

where ∇ is the gradient w.r.t g.

Proof. Let $x \in M \backslash N_1$. Choose $y \in N_1$, $u \in (T|_y N_1)^\perp$ such that $\gamma_u : [0, d_i(x)] \to M$ is the minimal normal unit speed geodesic from N_1 to x. Since $(0,1) \in \mathcal{D}_1|_y$ (because $y \in N_1$), the parallel translate of $(0,1)$ along γ_u stays in $\mathcal{D}_1$ which, in view of (21), gives

$$\begin{aligned}\mathcal{D}_1|_x \ni (P^{\nabla^{\mathsf{Rol}}})_0^{d_1(x)}(\gamma_u)(0,1) =&\big(-\sin(d_1(x))\dot\gamma_u(d_1(x)), \cos(d_1(x))\big)\\ =&\big(-\sin(d_1(x))\nabla(d_1(\cdot))|_x, \cos(d_1(x))\big)\\ =&\big(\nabla\cos(d_1(\cdot))|_x, \cos(d_1(x))\big),\end{aligned}$$

where the last two equalities hold true if x is not in the cut nor the conjugate locus of N_1 (nor is x in N_1, by assumption). Working in the complement of these points, which is a dense subset of M and using a continuity argument, we may assure that the result holds true everywhere where d_i is smooth. The same argument proves the formula (22) for d_2. □

Lemma 4.13 For every $Y \in \mathrm{VF}(M)$, one has, wherever $d_1(\cdot)$ is smooth, that

$$g\big(R(Y, \nabla d_1(\cdot))\nabla d_1(\cdot), Y\big) = g(Y,Y) - \big(\nabla_Y(d_1(\cdot))\big)^2. \tag{23}$$

Proof. It is known (see [23]) that for any $Y, Z \in \mathrm{VF}(M)$, $d_1(\cdot)$ satisfies a PDE

$$\begin{aligned}-g(R(Y|_y, \nabla d_1(y))\nabla d_1(y), Z|_y) =&\mathrm{Hess}^2(d_1(\cdot))(Y|_y, Z|_y)\\ &+\big(\nabla_{\nabla d_1(y)}\mathrm{Hess}(d_1(\cdot))\big)(Y|_y, Z|_y),\end{aligned}$$

for every $y \in M$ such that d_1 is smooth at y (and this is true in a dense subset of M). In particular, $y \notin N_1$. Also, since the set of points $y \in M$ where $\cos(d_1(y)) = 0$ or $\sin(d_1(y)) = 0$ is clearly Lebesgue zero-measurable, we may assume that $\cos(d_1(y)) \neq 0$ and $\sin(d_1(y)) \neq 0$.

Notice that $(X_0, r_0) := (\nabla\cos(d_1(\cdot)), \cos(d_1(\cdot)))$ belongs to $\mathcal{D}_1$ and has h-norm equal to 1. We may choose in a neighbourhood U of y vector fields $X_1,\dots,X_m \in \mathrm{VF}(U)$ such that (X_0, r_0), $(X_1, 0)$, …, $(X_m, 0)$ is an h-orthonormal basis of $\mathcal{D}_1$ over U. Assume also that (X_0, r_0) is smooth on U. This implies that there are smooth one-forms ω_j^i, $i,j = 0,\dots,m$ defined by (set here $r_1 = \cdots = r_m = 0$) $\nabla_Y^{\mathsf{Rol}}(X_i, r_i) = \sum_{i=0}^m \omega_i^j(Y)(X_j, r_j), \quad Y \in \mathrm{VF}(M)$, or, more explicitly,

$$\begin{cases}\nabla_Y X_j + r_j Y = \displaystyle\sum_{i=0}^m \omega_j^i(Y) X_i\\ Y(r_j) - g(Y, X_j) = \displaystyle\sum_{i=0}^m \omega_j^i(Y) r_i,\end{cases}$$

Since $(X_0, r_0), \dots, (X_m, r_m)$ are h-orthonormal, it follows that $\omega_j^i = -\omega_i^j$. The fact that $r_1 = \cdots = r_m = 0$ implies that $-g(Y, X_j) = \omega_j^0(Y) r_0, \quad j = 1, \dots, m$ i.e.,

$$\omega_0^j(Y) = \frac{g(Y, X_j)}{\cos(d_1(\cdot))}.$$

Since $\omega_0^0 = 0$, one has $\nabla_Y X_0 + r_0 Y = \sum_{j=1}^m \omega_0^j(Y) X_j$, which simplifies to

$$\nabla_Y \nabla d_1(\cdot) = -\cot(d_1(\cdot)) \nabla_Y (d_1(\cdot)) \nabla d_1(\cdot) + \cot(d_1(\cdot)) Y - \frac{1}{\sin(d_1(\cdot)) \cos(d_1(\cdot))} \sum_{j=1}^m g(X_j, Y) X_j.$$

Writing $S(Y) := \nabla_Y \nabla d_1(\cdot) = \mathrm{Hess}(d_1(\cdot))(Y, \cdot)$, one obtains

$$\begin{aligned}
&(\nabla_{\nabla d_1(\cdot)} S)(Y) = \nabla_{\nabla d_1(\cdot)} (S(Y)) - S(\nabla_{\nabla d_1(\cdot)} Y) \\
=&\frac{1}{\sin^2(d_1(\cdot))} \nabla_Y (d_1(\cdot)) \nabla d_1(\cdot) - \cot(d_1(\cdot)) g(\nabla_{\nabla d_1(\cdot)} Y, \nabla d_1(\cdot)) \nabla d_1(\cdot) \\
&- \frac{1}{\sin^2(d_1(\cdot))} Y - \Big(\frac{1}{\cos^2(d_1(\cdot))} - \frac{1}{\sin^2(d_1(\cdot))} \Big) \sum_{j=1}^m g(Y, X_j) X_j \\
&- \frac{1}{\sin(d_1(\cdot)) \cos(d_1(\cdot))} \sum_{j=1}^m \big(g(Y, \nabla_{\nabla d_1(\cdot)} X_j) X_j + g(Y, X_j) \nabla_{\nabla d_1(\cdot)} X_j \big) \\
&+ \cot(d_1(\cdot)) \underbrace{\nabla_{\nabla_{\nabla d_1(\cdot)} Y} (d_1(\cdot))}_{= g(\nabla d_1(\cdot), \nabla_{\nabla d_1(\cdot)} Y)} \nabla d_1(\cdot),
\end{aligned}$$

where we used that $\nabla_{\nabla d_1(\cdot)} (d_1(\cdot)) = g(\nabla d_1(\cdot), \nabla d_1(\cdot)) = 1$. On the other hand,

$$\begin{aligned}
&\mathrm{Hess}^2(d_1(\cdot))(Y, \cdot) = S^2(Y) = S(S(Y)) \\
=&S\Big(-\cot(d_1(\cdot)) \nabla_Y (d_1(\cdot)) \nabla d_1(\cdot) + \cot(d_1(\cdot)) Y - \frac{1}{\sin(d_1(\cdot)) \cos(d_1(\cdot))} \sum_{j=1}^m g(X_j, Y) X_j \Big) \\
=& -\cot^2(d_1(\cdot)) \nabla_Y (d_1(\cdot)) \nabla d_1(\cdot) + \cot^2(d_1(\cdot)) Y - \frac{2}{\sin^2(d_1(\cdot))} \sum_{j=1}^m g(X_j, Y) X_j \\
&+ \frac{1}{\sin^2(d_1(\cdot)) \cos^2(d_1(\cdot))} \sum_{j=1}^m g(X_j, Y) X_j,
\end{aligned}$$

where we used that $\nabla d_1(\cdot), X_1, \dots, X_m$ are g-orthonormal (recall that $X_0 = -\sin(d_1(\cdot)) \nabla d_1(\cdot)$.) Thus, for any $Y, Z \in \mathrm{VF}(M)$, one has on U that

$$\begin{aligned}
&-g(R(Y, \nabla d_1(\cdot)) \nabla d_1(\cdot), Z) = -g(Y, Z) + \Big(\frac{1}{\sin^2(d_1(\cdot))} - \cot^2(d_1(\cdot)) \Big) \nabla_Y (d_1(\cdot)) \nabla_Z (d_1(\cdot)) \\
&- \frac{1}{\sin(d_1(\cdot)) \cos(d_1(\cdot))} \sum_{j=1}^m \big(g(Y, \nabla_{\nabla d_1(\cdot)} X_j) g(X_j, Z) + g(Y, X_j) g(\nabla_{\nabla d_1(\cdot)} X_j, Z) \big).
\end{aligned}$$

We also set $Z = Y$ and hence get that $-g(R(Y, \nabla d_1(\cdot)) \nabla d_1(\cdot), Y)$ is equal to

$$g(Y, Y) - \nabla_Y (d_1(\cdot)) \nabla_Y (d_1(\cdot)) + \frac{2}{\sin(d_1(\cdot)) \cos(d_1(\cdot))} \sum_{j=1}^m g(Y, \nabla_{\nabla d_1(\cdot)} X_j) g(X_j, Y).$$

Here $\sum_{j=1}^m g(Y, \nabla_{\nabla d_1(\cdot)} X_j) g(X_j, Y)$ is equal to

$$
\begin{aligned}
&-\frac{1}{\sin(d_1(\cdot))} \sum_{j=1}^m g(Y, \nabla_{X_0} X_j) g(X_j, Y) = -\frac{1}{\sin(d_1(\cdot))} \sum_{j=1}^m g(Y, \sum_{i=1}^m \omega_j^i(X_0) X_i g(X_j, Y) \\
=&-\frac{1}{\sin(d_1(\cdot))} \sum_{i,j=1}^m \underbrace{\omega_j^i(X_0)}_{(\star)_1} \underbrace{g(Y, X_i) g(X_j, Y)}_{(\star)_2} = 0,
\end{aligned}
$$

where expression $(\star)_1$ is skew-symmetric in (i, j) while $(\star)_2$ is symmetric on (i, j). Hence the sum is zero. We finally obtain $g(R(Y, \nabla d_1(\cdot))\nabla d_1(\cdot), Y) = g(Y, Y) - \big(\nabla_Y(d_1(\cdot))\big)^2$, as claimed. It is clear that this formula now holds at every point of M where $d_1(\cdot)$ is smooth and for any $Y \in \mathrm{VF}(M)$. In particular, if Y is a unit vector g-perpendicular to $\nabla d_1(\cdot)$ at a point y of M, then $\nabla_Y d_1(\cdot)|_y = g(\nabla d_1(\cdot)|_y, Y|_y) = 0$ and hence $\sec(Y, d_1(\cdot))|_y = +1$. □

Lemma 4.14 For every $x \in N_1$, a unit vector $u \in (T|_x N_1)^\perp$ and $v \in T|_x M$ with $v \perp u$,

$$
\left\|(\exp_x)_*|_{tu}(v)\right\|_g = \left|\frac{\sin(t)}{t}\right| \|v\|_g, \quad t \in \mathbb{R}. \tag{24}
$$

In particular, for all unit vectors $u_1, u_2 \in (T|_x N_1)^\perp$ one has $\exp_x(\pi u_1) = \exp_x(\pi u_2)$.

Proof. Let $Y_{u,v}(t) = \frac{\partial}{\partial s}|_0 \exp_x(t(u + sv))$ be the Jacobi field along $\gamma_u(t) = \exp_x(tu)$ such that $Y_{u,v}(0) = 0$, $\nabla_{\dot\gamma_u(0)} Y_{u,v} = v$. Since $v \perp u$, it follows from the Gauss lemma (see [24]) that $Y_{u,v}(t) \perp \dot\gamma_u(t)$ for all t. Moreover, the assumption $u \in (T|_x N_1)^\perp$ implies that $\nabla d_1(\cdot)|_{\gamma_u(t)} = \dot\gamma_u(t)$ and thus $\nabla_{Y_{u,v}(t)}(d_1(\cdot)) = g(\dot\gamma_u(t), Y_{u,v}(t)) = 0$. By polarization, (23) rewrites as $R(Z(t), \dot\gamma_u(t))\dot\gamma_u(t) = Z(t) - g(Z(t), \dot\gamma_u(t))\dot\gamma_u(t)$, for any vector field Z along γ_u. In particular, $\nabla_{\dot\gamma_u} \nabla_{\dot\gamma_u} Y_{u,v} = -R(Y_{u,v}, \dot\gamma_u)\dot\gamma_u = -Y_{u,v}$, since
$g(Y_{u,v}(t), \dot\gamma_u(t)) = 0$ for all t. On the other hand, the vector field $Z(t) = \sin(t) P_0^t(\gamma_u) v$ satisfies along γ_u, for all t that $\nabla_{\dot\gamma_u(t)} \nabla_{\dot\gamma_u} Z = -Z(t)$ with $Z(0) = 0$ and $\nabla_{\dot\gamma_u} Z|_{t=0} = v$, i.e., the same initial value problem as $Y_{u,v}$. This implies that $Y_{u,v}(t) = \sin(t) P_0^t(\gamma_u) v$, from which we obtain (24) because $Y_{u,v}(t) = t(\exp_x)_*|_{tu}(v)$. The last claim follows from the fact that the map $\exp_x|_S : S \to M$ where $S = \{u \in (T|_x N_1)^\perp \mid \|u\| = \pi\}$ is a constant map. Indeed, if $u \in S$, $v \in T|_u S$ and we identify v as an element of $T|_x M$ as usual, then by what we have just proved (note that $u = \pi \frac{u}{\|u\|_g}$), $\|(\exp_x)_*|_u(v)\|_g = \frac{\sin(\pi)}{\pi} \|v\|_g = 0$. Hence $\exp_x|_S$ has zero differential on all over S which is connected, since its dimension is $n - m - 1 \geq 1$ by assumption. Hence $\exp_x|_S$ is a constant map.

□

Lemma 4.15 For every $x \in N_1$ and unit normal vector $u \in (T|_x N_1)^\perp$, the geodesic $t \mapsto \gamma_u(t)$ meets N_2 exactly at $t \in (\mathbb{Z} + \frac{1}{2})\pi$, similarly with the roles of N_1 and N_2 interchanged.

Proof. Let $x \in N_1$ and $u \in (T|_x N_1)^\perp$ be a unit vector normal vector to N_1. For $(X, r) \in \mathcal{D}_1|_x$ define $(X(t), r(t)) = (P^{\nabla^{\mathrm{Rol}}})_0^t(\gamma_u)(X, r)$. Then by (16), (17) we have (notice that $g(u, X) = 0$ since $u \in (T|_x N_1)^\perp = (\mathcal{D}_1^M|_x)^\perp$ and $X \in \mathcal{D}_1^M|_x$) $r(t) = r(0)\cos(t)$. Hence, $(X(t), r(t))$ is h-orthogonal to $(0, 1)$ if and only of $r(t) = 0$ i.e., $r(0)\cos(t) = 0$. This proves that $(0, 1) \perp \mathcal{D}_1|_{\gamma_u(t)}$ i.e., $(0, 1) \in \mathcal{D}_2|_{\gamma_u(t)}$ i.e., $\gamma_u(t) \in N_2$ if and only if $t \in (\frac{1}{2} + \mathbb{Z})\pi$ (obviously, there is a vector $(X, r) \in \mathcal{D}_1|_x$ with $r \neq 0$).

□

Lemma 4.16 The submanifolds N_1, N_2 are isometrically covered by Euclidean spheres of dimensions m and $n - m$, respectively, and the fundamental groups of N_1 and N_2 are finite and have the same number of elements. More precisely, for any $x \in N_1$ define

$$S_x = \{u \in (T|_x N_1)^\perp \mid \|u\|_g = 1\},$$

equipped with the restriction of the metric $g|_x$ of $T|_x M$. Then $S_x \to N_2; \quad u \mapsto \exp_x(\frac{\pi}{2}u)$, is a Riemannian covering. The same claim holds with N_1 and N_2 interchanged.

Proof. Denote by C_1 the component of N_1 containing x. We show first that $C_1 = N_1$ i.e., N_1 is connected. Let $y_1 \in N_1$. Since C_1 is a closed subset of M, there is a minimal geodesic γ_v in M from C_1 to y_1 with $\dot\gamma_v(0) = v$ a unit vector, $x_1 := \gamma_v(0) \in C_1$ and $\gamma_v(d) = y_1$, with $d := d_g(y_1, C_1)$. By minimality, $v \in (T|_{x_1}C_1)^\perp = (T|_{x_1}N_1)^\perp$. Hence by Lemma 4.15, $x_2 := \exp_{x_1}(\frac{\pi}{2}v) = \gamma_v(\frac{\pi}{2})$ belongs to N_2. Since the set $S_{x_2} = \{u \in (T|_{x_2}N_2)^\perp \mid \|u\|_g = 1\}$ is connected (its dimension is $m \geq 1$ by assumption), Lemma 4.15 implies that $\exp_{x_2}\big(\frac{\pi}{2}S_{x_2}\big)$ is contained in a single component C_1' of N_1. Writing $u := \dot\gamma_v(\frac{\pi}{2})$, we have $\pm u \in S_{x_2}$ so

$$C_1' \ni \exp_{x_2}(-\tfrac{\pi}{2}u) = \exp_{x_2}\big(-\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}} \exp_{x_1}(tv)\big) = \exp_{x_1}((\tfrac{\pi}{2} - t)v)|_{t=\frac{\pi}{2}} = x_1,$$

and since also $x_1 \in C_1$, it follow that $C_1' = C_1$. But this implies that

$$\gamma_v(\pi) = \exp_{x_1}(\pi v) = \exp_{x_2}\big(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}} \exp_{x_1}(tv)\big) = \exp_{x_2}(\tfrac{\pi}{2}u) \in C_1.$$

It also follows from $u \in (T|_{x_2}N_2)^\perp$ that $\dot\gamma_v(\pi) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}} \exp_{x_2}(tu) \in (T|_{\gamma_v(\pi)}N_1)^\perp$. Since $\exp_{x_2}((d - \frac{\pi}{2})u) = y_1 \in N_1$, Lemma 4.15 implies that $d - \frac{\pi}{2} \in (\frac{1}{2} + \mathbb{Z})\pi$, from which, since $d \geq 0$, we get $d \in \mathbb{N}_0\pi$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. By taking $x_2' = \gamma_v(\frac{3}{2}\pi) \in N_2$ we may show similarly that $\gamma_v(2\pi) \in C_1$ and by induction we get $\gamma_v(k\pi) \in C_1$ for every $k \in \mathbb{N}_0$. In particular, since $d \in \mathbb{N}_0\pi$, we get $y_1 = \gamma_v(d) \in C_1$. Since $y_1 \in N_1$ was arbitrary, we get $N_1 \subset C_1$ which proves the claim. Repeating the argument with N_1 and N_2 interchanged, we see that N_2 is connected.

Eq. (24) shows that, taking $u \in S_x$ and $v \in T|_u S_x$, i.e., $v \perp u$, $v \perp T|_x N_1$,

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \exp_x\big(\frac{\pi}{2}(u + tv)\big)\Big\|_g = \Big\|(\exp_x)_*|_{\frac{\pi}{2}u}(\frac{\pi}{2}v)\Big\|_g = \|v\|_g\,.$$

This shows that $u \mapsto \exp_x(\frac{\pi}{2}u)$ is a local isometry $S_x \to N_2$. In particular, the image is open and closed in N_2, which is connected, hence $u \mapsto \exp_x(\frac{\pi}{2}u)$ is onto N_2. According to Proposition II.1.1 in [24], $u \mapsto \exp_x(\frac{\pi}{2}u)$ is a covering $S_x \to N_2$.

Similarly, for any $y \in N_2$ the map $S_y \to N_1$; $u \mapsto \exp_y(\frac{\pi}{2}u)$ is a Riemannian covering.

Finally, let us prove the statement about fundamental groups. Fix a point $x_i \in N_i$ and write $\phi_i(u) = \exp_{x_i}(\frac{\pi}{2}u)$, $i = 1, 2$, for maps $\phi_1 : S_{x_1} \to N_2$, $\phi_2 : S_{x_2} \to N_1$. The fundamental groups $\pi_1(N_1)$, $\pi_1(N_2)$ of N_1, N_2 are finite since their universal coverings are the (normal) spheres S_{x_2}, S_{x_1} which are compact. Also, $\phi_1^{-1}(x_2)$ and $\phi_2^{-1}(x_1)$ are in one-to-one correspondence with $\pi_1(N_2)$ and $\pi_1(N_1)$ respectively.

Define $\Phi_1 : \phi_1^{-1}(x_2) \to S_{x_2}$; $\Phi_1(u) = -\frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}} \exp_{x_1}(tu) \in S_{x_2}$ and similarly $\Phi_2 : \phi_2^{-1}(x_1) \to S_{x_1}$; $\Phi_2(u) = -\frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}} \exp_{x_2}(tu) \in S_{x_1}$. Clearly, for $u \in \phi_1^{-1}(x_2)$,

$$\phi_2(\Phi_1(u)) = \exp_{x_2}\big(-\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}} \exp_{x_1}(tu)\big) = \exp_{x_1}((\tfrac{\pi}{2} - t)u)|_{t=\frac{\pi}{2}} = x_1,$$

i.e., Φ_1 maps $\phi_1^{-1}(x_2) \to \phi_2^{-1}(x_1)$. Similarly Φ_2 maps $\phi_2^{-1}(x_1) \to \phi_1^{-1}(x_2)$. Finally, Φ_1 and Φ_2 are inverse maps to each other since for $u \in \phi_1^{-1}(x_2)$,

$$\Phi_2(\Phi_1(u)) = -\frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}} \exp_{x_2}\big(-t\frac{\mathrm{d}}{\mathrm{d}s}\big|_{\frac{\pi}{2}} \exp_{x_1}(su)\big) = -\frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}} \exp_{x_1}((\tfrac{\pi}{2} - t)u) = u,$$

and similarly $\Phi_1(\Phi_2(u)) = u$ for $u \in \phi_2^{-1}(x_1)$.

□

For sake of simplicity, we finish the proof of Theorem 4.8 under the assumption that N_2 is simply connected and indicate in Remark 4.19 below how to handle the general case.

If N_2 is simply connected, then $S_x \to N_2; \quad u \mapsto \exp_x(\frac{\pi}{2}u)$, defined in Lemma 4.16 is an isometry for some (and hence every) $x \in N_1$. From Lemma 4.16, it follows that N_1 is (simply connected and) isometric to a sphere as well. We next get the following.

Lemma 4.17 Fix $x_i \in N_j$, $j = 1, 2$ and let

$$S_{x_1} = \{u \in (T|_{x_1}N_1)^\perp \mid \|u\|_g = 1\}, \qquad S_{x_2} = \{u \in (T|_{x_2}N_2)^\perp \mid \|u\|_g = 1\},$$

the unit normal spheres to N_1, N_2 at x_1, x_2 respectively. Consider first the maps

$$\begin{aligned} f_1 : S_{x_1} &\to N_2 & f_2 : S_{x_2} &\to N_1 \\ f_1(u) &= \exp_{x_1}(\frac{\pi}{2}u) & f_2(v) &= \exp_{x_2}(\frac{\pi}{2}v), \end{aligned} \tag{25}$$

and the map w which associates to $(u, v) \in S_{x_1} \times S_{x_2}$ the unique element of $S_{f_2(v)}$ such that $\exp_{f_2(v)}(\frac{\pi}{2}w(u, v)) = f_1(u)$. Finally let

$$\begin{aligned} \Psi &:]0, \frac{\pi}{2}[\times S_{x_1} \times S_{x_2} \to M \\ \Psi(t, u, v) &= \exp_{f_2(v)}(tw(u, v)). \end{aligned} \tag{26}$$

Suppose that $\tilde{S} :=]0, \frac{\pi}{2}[\times S_{x_1} \times S_{x_2}$ is endowed with the metric $\tilde{g}$ such that

$$\tilde{g}|_{(t,u,v)} = \mathrm{d}t^2 + \sin^2(t) g|_{T|_u S_{x_1}} + \cos^2(t) g|_{T|_v S_{x_2}}.$$

Then Ψ is a local isometry.

Proof. We use G to denote the *geodesic vector field* on TM i.e., for $u \in TM$ we have

$$G|_u := \ddot{\gamma}_u(0) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big|_0 \exp_{\pi_{TM}(u)}(tu).$$

The projections on M by π_{TM} of its integral curves are geodesics. Indeed, first we notice that

$$G|_{\dot{\gamma}_u(t)} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\big|_0 \exp_{\gamma_u(t)}(s\dot{\gamma}_u(t)) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\big|_0 \gamma_u(t + s) = \ddot{\gamma}_u(t),$$

and hence, if Γ be a curve on TM defined by $\Gamma(t) = \dot{\gamma}_u(t)$, then $\dot{\Gamma}(t) = \ddot{\gamma}_u(t) = G|_{\dot{\gamma}_u(t)} = G|_{\Gamma(t)}$, and $\Gamma(0) = u$. Hence Γ satisfies the same initial value problem as $t \mapsto \Phi_G(t, u)$, which implies that $\Phi_G(t, u) = \dot{\gamma}_u(t), \quad \forall t \in \mathbb{R},\ u \in TM$, and in particular, $(\pi_{TM} \circ \Phi_G)(t, u) = \gamma_u(t), \quad \forall t \in \mathbb{R},\ u \in TM$.

For every $u \in TM$ there is a direct sum decomposition $H_u \oplus V_u$ of $T|_u TM$ where $V_u = V|_u(\pi_{TM})$ is the π_{TM}-vertical fiber over u and H_u is defined as

$$H_u = \{\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 P_0^t(\gamma_X)u \mid X \in T|_{\pi_{TM}(u)}M\}.$$

We write the elements of $T|_u TM$ w.r.t. this direct sum decomposition as (A, B) where $A \in H_u$, $B \in V_u$. It can now be shown that (see [24] Lemma 4.3, Chapter II)

$$((\Phi_G)_t)_*|_u(A, B) = (Z_{(A,B)}(t), \nabla_{\dot{\gamma}_u(t)} Z_{(A,B)}), \quad (A, B) \in T|_u TM, \quad u \in TM,$$

with $Z_{(A,B)}$, the unique Jacobi field along geodesic γ_u such that $Z_{(A,B)}(0) = A$, $\nabla_{\dot{\gamma}_u(0)} Z_{(A,B)} = B$.

We are now ready to prove the claim. First observe that $\Psi(t,u,v)=(\pi_{TM}\circ\Phi_G)(t,w(u,v))$ and hence, for $(\frac{\partial}{\partial t},X_1,X_2)\in T\tilde{S}$,

$$\begin{aligned}\Psi_*(\frac{\partial}{\partial t},X_1,X_2)=&(\pi_{TM})_*\big(\frac{\partial}{\partial t}\Phi_G(t,w(u,v))+((\Phi_G)_t)_*|_{w(u,v)}w_*(X_1,X_2)\big)\\ =&(\pi_{TM})_*\Big(G|_{\Phi_G(t,w(u,v))}+\big(Z_{w_*(X_1,X_2)}(t),\nabla_{\frac{\partial}{\partial t}(\pi_{TM}\circ\Phi_G)(t,w(u,v))}Z_{w_*(X_1,X_2)}\big)\Big)\\ =&\dot\gamma_{w(u,v)}(t)+Z_{w_*(X_1,X_2)}(t).\end{aligned}$$

On the other hand, $(\pi_{TM}\circ\Phi_G)\big(\frac{\pi}{2},w(u,v)\big)=f_1(u)$, from where $(f_1)_*|_u(X_1)=Z_{w_*(X_1,X_2)}\big(\frac{\pi}{2}\big)$. Similarly, since $(\pi_{TM}\circ\Phi_G)\big(0,w(u,v)\big)=\pi_{TM}(w(u,v))=f_2(v)$, we get $(f_2)_*|_v(X_2)=Z_{w_*(X_1,X_2)}(0)$.

As in the proof of Lemma 4.14, we see that the Jacobi equation that $Z_{w_*(X_1,X_2)}$ satisfies is $\nabla_{\dot\gamma_{w(u,v)}(t)}\nabla_{\dot\gamma_{w(u,v)}}Z_{w_*(X_1,X_2)}=-Z_{w_*(X_1,X_2)}(t)$. It is clear that this implies that $Z_{w_*(X_1,X_2)}$ has the form $Z_{w_*(X_1,X_2)}(t)=\sin(t)P_0^t(\gamma_{w(u,v)})V_1+\cos(t)P_0^t(\gamma_{w(u,v)})V_2$, for some $V_1,V_2\in T|_{f_2(u)}M$. Using the boundary values of $Z_{w_*(X_1,X_2)}(t)$ at $t=0$ and $t=\frac{\pi}{2}$ as derived above, we get $V_1=P^0_{\frac{\pi}{2}}(\gamma_{w(u,v)})((f_1)_*|_u(X_1))$ and $V_2=(f_2)_*|_v(X_2)$. Define

$$\begin{aligned}Y_1(t)&=\sin(t)P_0^t(\gamma_{w(u,v)})V_1=\sin(t)P^t_{\frac{\pi}{2}}(\gamma_{w(u,v)})((f_1)_*|_u(X_1)),\\ Y_2(t)&=\cos(t)P_0^t(\gamma_{w(u,v)})V_2=\cos(t)P_0^t(\gamma_{w(u,v)})((f_2)_*|_v(X_2)),\end{aligned}$$

which means that $Z=Y_1+Y_2$. Notice that Y_1 and Y_2 are Jacobi fields along $\gamma_{w(u,v)}$.

Since $w(u,v)\in(T|_{f_2(v)}N_1)^\perp$ and $\dot\gamma_{w(u,v)}(\frac{\pi}{2})\in(T|_{f_1(u)}N_2)^\perp$ and

$$Y_1(\tfrac{\pi}{2})=(f_1)_*|_u(X_1)\in T|_{f_1(u)}N_2,\quad Y_2(0)=(f_2)_*|_v(X_2)\in T|_{f_2(v)}N_1,$$

it follows that $Y_1,Y_2\perp\gamma_{w(u,v)}$. We claim that moreover $Y_1\perp Y_2$. Indeed, since $(f_2)_*|_v(X_2)\in T|_{f_2(v)}N_1$ and $(0,1)\in\mathcal{D}_1|_{f_2(v)}$ (by definition of N_1), we have $((f_2)_*|_v(X_2),0)\in\mathcal{D}_1|_{f_2(v)}$ and hence, for all t, $(Z_1(t),r_1(t)):=(P^{\nabla^{\mathsf{Rol}}})_0^t(\gamma_{w(u,v)})((f_2)_*|_v(X_2),0)\in\mathcal{D}_1$. On the other hand, r_1 satisfies $\ddot r_1+r_1=0$ with initial conditions $r_1(0)=0$ and
$\dot r_1(0)=g(\dot\gamma_{w(u,v)}(0),Z_1(0))=g(w(u,v),(f_2)_*|_v(X_2))=0$ so $r_1(t)=0$ for all t. Thus $Z_1(t)$ satisfies $\nabla_{\dot\gamma_{w(u,v)}(t)}Z_1=0$ i.e., $Z_1(t)=P_0^t(\gamma_{w(u,v)})((f_2)_*|_v(X_2))$. Similarly, if
$w'(u,v):=-\frac{\mathrm{d}}{\mathrm{d}t}\big|_{\frac{\pi}{2}}\exp_{f_2(v)}(tw(u,v))=-\dot\gamma_{w(u,v)}(\frac{\pi}{2})$,

$$(Z_2(\tfrac{\pi}{2}-t),r_2(\tfrac{\pi}{2}-t)):=(P^{\nabla^{\mathsf{Rol}}})_0^t(\gamma_{w'(u,v)})((f_1)_*|_u(X_1),0)\in\mathcal{D}_2,$$

and we have $r_2(\frac{\pi}{2}-t)=0$ and $Z_2(\frac{\pi}{2}-t)=P_0^t(\gamma_{w'(u,v)})((f_1)_*|_v(X_1))$ i.e., $Z_2(t)=P^t_{\frac{\pi}{2}}(\gamma_{w(u,v)})((f_1)_*|_v(X_1))$. But since $\mathcal{D}_1\perp\mathcal{D}_2$ w.r.t. h, we have that $(Z_1,r_1)\perp(Z_2,r_2)$ w.r.t. h i.e., $g(Z_1(t),Z_2(t))=0$ for all t (since $r_1(t)=r_2(t)=0$). Thus,

$$\begin{aligned}g(Y_1(t),Y_2(t))=&\sin(t)\cos(t)g\big(P^t_{\frac{\pi}{2}}(\gamma_{w(u,v)})((f_1)_*|_u(X_1)),P_0^t(\gamma_{w(u,v)})((f_2)_*|_v(X_1))\big)\\ =&\sin(t)\cos(t)g(Z_2(t),Z_1(t))=0\end{aligned}$$

This proves the claim, i.e., $Y_1\perp Y_2$. Since $\|w(u,v)\|_g=1$, one has

$$\begin{aligned}\Big\|\Psi_*(\frac{\partial}{\partial t},X_1,X_2)\Big\|_g^2=&\big\|\dot\gamma_{w(u,v)}(t)+Y_1(t)+Y_2(t)\big\|_g^2=\big\|\dot\gamma_{w(u,v)}(t)\big\|_g^2+\|Y_1(t)\|_g^2+\|Y_2(t)\|_g^2\\ =&1+\sin^2(t)^2\,\|(f_1)_*|_u(X_1)\|_g^2+\cos^2(t)\,\|(f_2)_*|_v(X_2)\|_g.\end{aligned}$$

Finally, since $(f_1)_*|_u(X_1)=(\exp_{x_1})_*|_{\frac{\pi}{2}u}(\frac{\pi}{2}X_1)$ and $(f_2)_*|_v(X_2)=(\exp_{x_2})_*|_{\frac{\pi}{2}v}(\frac{\pi}{2}X_2)$, Eq. (24) implies that $\|(f_1)_*|_u(X_1)\|_g=|\sin(\frac{\pi}{2})|\,\|X_1\|_g=\|X_1\|_g$, $\|(f_2)_*|_v(X_2)\|_g=|\sin(\frac{\pi}{2})|\,\|X_2\|_g=\|X_2\|_g$, and therefore $\big\|\Psi_*\big(\frac{\partial}{\partial t},X_1,X_2\big)\big\|_g^2=1+\sin^2(t)\,\|X_2\|_g^2+\cos^2(t)\,\|X_1\|_g^2=\tilde g|_{(t,u,v)}\big(\frac{\partial}{\partial t},X_1,X_2\big)$ i.e., Ψ is a local isometry $\tilde S\to M$. □

Lemma 4.18 The manifold M has constant constant curvature equal to 1.

Proof. By Lemma 4.17, we know that $\Psi : \tilde{S} \to M$ is a local isometry. Now $(\tilde{S}, \tilde{g})$ has constant curvature $= 1$ since it is isometric to an open subset of the unit sphere (cf. [23] Chapter 1, Section 4.2). The image $\Psi(\tilde{S})$ of Ψ is clearly a dense subset of M (indeed, $\Psi(\tilde{S}) = M \backslash (N_1 \cup N_2)$), which implies that M has constant curvature $= 1$. □

This completes the proof the theorem in the case $1 \leq m \leq n-2$, since a complete Riemannian manifold (M, g) with constant curvature $= 1$ is covered, in a Riemannian sense, by the unit sphere i.e., S^n. The cases $m = 0$ and $m = n-1$ i.e., $\dim \mathcal{D}_1 = 1$ and $\dim \mathcal{D}_2 = 1$, respectively, are treated exactly in the same way as above, but in this case N_1 is a discrete set which might not be connected. □

Remark 4.19 The argument can easily be modified to deal with the case where N_2 (nor N_1) is not simply-connected. The simplifying assumption of simply connectedness of N_1 and N_2 made previously just serves to render the map $w(\cdot,\cdot)$ globally defined on $S_{x_1} \times S_{x_2}$. Otherwise we must define w only locally and, in its definition, make a choice corresponding to different sheets (of which there is a finite number).

4.6 Non-transitive Irreducible Action

Following the same line of arguments that have been used in proving the classification of Riemannian holonomy groups, the next step to take after proving Theorem 4.8 consists of studying the case where $H^{\mathsf{Rol}}|_{x_0}$ acts irreducibly on $T|_{x_0}M \oplus \mathbb{R}$ and non-transitively on the h_1-unit sphere of $T|_{x_0}M \oplus \mathbb{R}$, where the latter means that there are more than one $H^{\mathsf{Rol}}|_{x_0}$-orbit on that unit sphere. We will prove that in this case of irreducible and non-transitive action of the rolling holonomy group, the manifold (M, g) has to have, again, constant curvature one. To do this, we will use the results from [26].

For the ease of reading, we first recall some definitions and the key results from [26]. Let V be a vector space. The action of $\mathrm{GL}(V)$ on V induces in a natural way an action of $\mathrm{GL}(V)$ on the tensor spaces $\bigotimes^k V^* \otimes \bigotimes^m V$ of (k,m)-tensors by

$$(gT)(X_1, \ldots, X_k, \omega_1, \ldots, \omega_m) := T(g^{-1}X_1, \ldots, g^{-1}X_k, \omega_1 \circ g, \ldots, \omega_m \circ g),$$

where $T \in \bigotimes^k V^* \otimes \bigotimes^m V$, $X_1, \ldots, X_k \in V$, $\omega_1, \ldots, \omega_m \in V^*$.

If $\mathcal{P} \in \otimes^3 V^* \otimes V$, we write usually $\mathcal{P}(X,Y)Z$ for $\mathcal{P}(X,Y,Z,\cdot) \in V$, where $X, Y, Z \in V$. If $g \in \mathrm{GL}(V)$ and $\mathcal{P}$ is a $(1,3)$ tensor, then $(g\mathcal{P})(X,Y) = g \circ \mathcal{P}(g^{-1}X, g^{-1}Y) \circ g^{-1}$. This implies that $\mathfrak{gl}(V)$ acts on an element $\bigotimes^3 V^* \otimes V$ by

$$(A\mathcal{P})(X,Y) = -\mathcal{P}(AX,Y) - \mathcal{P}(X,AY) - [\mathcal{P}(X,Y), A]_{\mathfrak{so}}, \tag{27}$$

where $A \in \mathfrak{gl}(V)$. Let G be a subgroup of $O(V)$, where V is an inner product space. We recall that G acts (a) irreducibly in V if the only G-invariant subspaces of G are $\{0\}$ and V and (b) transitively on (the unit sphere of) V if for one (and hence any) unit vector $X \in V$ one has $GX = S^{n-1}(V)$, where $S^{n-1}(V)$ is the unit sphere of V. We also recall that if a connected subgroup G of $O(V)$ acts irreducibly in V, then G is compact (see [16], Appendix 5). The concept of a curvature tensor, in abstract setting, is defined as follows.

Definition 4.20 Let V be an vector space with inner product $\langle\cdot,\cdot\rangle$. Then a $(1,3)$-tensor $\mathcal{R} \in \otimes^3 V^* \otimes V$ is called *a curvature (tensor) in* V if the following conditions hold for all $X, Y, Z, W \in V$

$$\mathcal{R}(X,Y) = -\mathcal{R}(Y,X), \tag{28}$$
$$\langle \mathcal{R}(X,Y)Z, W\rangle = -\langle \mathcal{R}(X,Y)W, Z\rangle, \tag{29}$$
$$\langle \mathcal{R}(X,Y)Z, W\rangle = \langle \mathcal{R}(Z,W)X, Y\rangle, \tag{30}$$
$$\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y = 0,. \tag{31}$$

From these one makes the following observations. Eqs. (28), (29) imply that $\mathcal{R}$ can be seen as a map $\mathcal{R} : \wedge^2 V \to \wedge^2 V$ by defining $\langle \mathcal{R}(X \wedge Y), Z \wedge W\rangle$ to be $\langle \mathcal{R}(X,Y)Z, W\rangle$, where in the former $\langle\cdot,\cdot\rangle$ is the inner product in $\wedge^2 V$ induced in the standard way by $\langle\cdot,\cdot\rangle$ in V. Then Eq. (30) means that $\mathcal{R}$ as a map $\wedge^2 V \to \wedge^2 V$ is orthogonal.

Definition 4.21 Let V be an inner product space, G a compact subgroup of $O(V)$ with Lie algebra $\mathfrak{g}$ and $\mathcal{R}$ a curvature tensor in V. The triple $(V, \mathcal{R}, G)$ is a *holonomy system* if

$$\mathcal{R}(X,Y) \in \mathfrak{g}, \quad \forall X, Y \in V.$$

Notice that by (29), if $\mathcal{R}$ is a curvature in V, then $\mathcal{R}(X,Y) : V \to V$ is skew-symmetric i.e. $\mathcal{R}(X,Y) \in \mathfrak{so}(V)$ for all $X, Y \in V$. Moreover, it is easy to see that for all $g \in O(V)$ one has that $g\mathcal{R}$ is a curvature in V.

Definition 4.22 If $(V, \mathcal{R}, G)$ is a holonomy system, we write $G(\mathcal{R})$ for the linear span in $\mathfrak{gl}(V)$ of $\{(g\mathcal{R})(X,Y) \mid X, Y \in V\}$.

Clearly for all $\mathcal{Q} \in G(\mathcal{R})$, $g \in G$ one has $g\mathcal{Q} \in G(\mathcal{R})$ and hence $A\mathcal{Q} \in G(\mathcal{R})$ for all $A \in \mathfrak{g}$. Moreover, if $g \in G$ and $X, Y \in V$, then since one can write $(g\mathcal{R})(X,Y)$ as $\mathrm{Ad}(g)\mathcal{R}(X,Y)$ which belongs to $\mathfrak{g}$, because $\mathcal{R}(X,Y) \in \mathfrak{g}$, we get that $\mathcal{Q}(X,Y) \in \mathfrak{g}$ for all $\mathcal{Q} \in G(\mathcal{R})$, $X, Y \in V$. Thus we may pose the following definition.

Definition 4.23 If $(V, \mathcal{R}, G)$ is a holonomy system, we define $\mathfrak{g}^{\mathcal{R}}$ as the linear span of $\{\mathcal{Q}(X,Y) \mid \mathcal{Q} \in G(\mathcal{R}),\ X, Y \in V\}$ in $\mathfrak{g}$.

The subset $\mathfrak{g}^{\mathcal{R}}$ of $\mathfrak{g}$ is more than just a subspace as will be shown next.

Lemma 4.24 The linear space $\mathfrak{g}^{\mathcal{R}}$ is an ideal in $\mathfrak{g}$.

Proof. Let $\mathcal{Q} \in G(\mathcal{R})$, $X, Y \in V$, $A \in \mathfrak{g}$. By Eq. (27),

$$[\mathcal{Q}(X,Y), A]_{\mathfrak{so}} = -\mathcal{Q}(AX, Y) - \mathcal{Q}(X, AY) - (A\mathcal{Q})(X,Y).$$

We observed just before the previous definition that $A\mathcal{Q} \in G(\mathcal{R})$. Thus all the terms on the right belong to $\mathfrak{g}^{\mathcal{R}}$ by the very definition of it. Therefore $\mathfrak{g}^{\mathcal{R}}$ is an ideal in $\mathfrak{g}$. $\square$

Hence the following definition makes sense.

Definition 4.25 Let $(V, \mathcal{R}, G)$ be a holonomy system. We write $G^{\mathcal{R}}$ for the Lie-subgroup of G corresponding to the ideal $\mathfrak{g}^{\mathcal{R}}$ of $\mathfrak{g}$.

We need to define the concepts of a irreducible, transitive and symmetric holonomy systems.

Definition 4.26 A holonomy system $(V, \mathcal{R}, G)$ is said to be

1) *reducible* (resp. *irreducible*) if G acts reducibly (resp. irreducibly) in V;

2) *symmetric* if $g\mathcal{R} = \mathcal{R}$ for all $g \in G$.

If G is connected, the symmetry 2) of a holonomy system $(V, \mathcal{R}, G)$ can be written in the infinitesimal way as: $A\mathcal{R} = 0$, $\forall A \in \mathfrak{g}$. We state the main result of [26].

Proposition 4.27
- Let $(V, \mathcal{R}, G)$ be an irreducible holonomy system. If $G^{\mathcal{R}}$ does not act transitively on (the unit sphere of) V, then $(V, \mathcal{R}, G)$ is symmetric.
- If $(V, \mathcal{R}, G)$ and $(V, \mathcal{R}', G)$ are two irreducible symmetric holonomy systems with the same V and G and if both $\mathcal{R}$ and $\mathcal{R}'$ are non-zero, then there exists $c \in \mathbb{R}$ such that $\mathcal{R}' = c\mathcal{R}$.

We next deduce from the previous proposition our main result.

Theorem 4.28 Let (M, g) be a simply connected Riemannian manifold and (S^n, s_{n+1}) be the unit sphere. Then the rolling holonomy group $H^{\mathsf{Rol}}|_{x_0}$, for some $x_0 \in M$, cannot act both irreducibly and non-transitively on $T|_{x_0}M \oplus \mathbb{R}$.

Proof. We argue by contradiction. Assume that $H^{\mathsf{Rol}}|_{x_0}$ acts irreducibly and non-transitively on $T|_{x_0}M \oplus \mathbb{R}$. Since M is connected, it follows that for any $x \in M$, $H^{\mathsf{Rol}}|_x$ acts irreducibly and non-transitively on $T|_x M \oplus \mathbb{R}$. We will continue using x_0 in the notations below, but we don't consider it to be fixed anymore. Notice moreover that simply connectedness of M implies that $H^{\mathsf{Rol}}|_{x_0}$ is connected. Write $M \times \mathbb{R}$. The canonical, positively directed unit vector field in the $\mathbb{R}$ gives rise to a vector field ∂_t in $M \times \mathbb{R}$ in a natural way. We equip $M \times \mathbb{R}$ with the metric h_1,

$$h_1((X, r\partial_t), (Y, s\partial_t)) = g(X, Y) + rs, \quad (X, r\partial_t), (Y, s\partial_t) \in T(M \times \mathbb{R}).$$

If $\mathrm{pr}_1 : M \times \mathbb{R} \to M$ is the projection onto the first factor, then the pull-back bundle $\mathrm{pr}_1^*(\pi_{TM \oplus \mathbb{R}})$ is canonically isomorphic to $\pi_{T(M\times\mathbb{R})}$. We define a connection ∇^{R} on the manifold as the pull-back $\mathrm{pr}_1^*(\nabla^{\mathsf{Rol}})$ determined by

$$\begin{aligned}\nabla^{\mathrm{R}}_{(X, r\partial_t)}(Y, s\partial_t) =& \nabla^{\mathsf{Rol}}_X(Y, s), \quad \forall X, Y \in \mathrm{VF}(M),\ r, s \in C^\infty(M) \\ =& \big(\nabla_X Y + sX, (X(s) - g(X, Y))\partial_t\big).\end{aligned}$$

One has ∇^{R} is h_1-compatible (i.e. metric w.r.t. h_1) and if $T^R := T^{\nabla^{\mathrm{R}}}$, then $T^{\mathrm{R}}((X, r\partial_t), (Y, s\partial_t)) = r(Y, s\partial_t) - s(X, r\partial_t)$, so it is not the Levi-Civita connection of $(M \times \mathbb{R}, h_1)$.

Write $H^{\mathrm{R}} := H^{\nabla^{\mathrm{R}}}$ for the holonomy group(s) of ∇^{R}. Next we show that for every $(x_0, s_0) \in M \times \mathbb{R}$, one has $H^{\mathrm{R}}|_{(x_0,s_0)} = H^{\mathsf{Rol}}|_{x_0}$, where the isomorphism $T|_{(x_0,s_0)}(M \times \mathbb{R}) \cong T|_{x_0}M \oplus \mathbb{R}$ is understood. Indeed, suppose $(\gamma, \tau) : [0, 1] \to M \times \mathbb{R}$ is a piecewise smooth path, $(\gamma, \tau)(0) = (x_0, s_0)$ and $(X_0, r_0\partial_t|_{s_0}) \in T|_{(x_0,s_0)}(M \times \mathbb{R})$. Let $(X(t), r(t)\partial_t|_{\tau(t)}) := (P^{\nabla^{\mathrm{R}}})_0^t(\gamma, \tau)(X_0, r_0\partial_t|_{s_0})$ and $(\overline{X}(t), \overline{r}(t)) := (P^{\nabla^{\mathsf{Rol}}})_0^t(\gamma)(X_0, r_0)$. It is enough to show that $(\overline{X}(t), \overline{r}(t)\partial_t|_{\tau(t)}) = (X(t), r(t)\partial_t|_{\tau(t)})$ for all $t \in [0, 1]$. But this is clear since $\nabla^{\mathrm{R}}_{(\dot\gamma(t), \dot\tau(t)\partial_t|_{\tau(t)})}(\overline{X}, \overline{r}\partial_t) = \nabla^{\mathsf{Rol}}_{\dot\gamma(t)}(\overline{X}, \overline{r}) = 0$. Thus for every $(x_0, s_0) \in M \times \mathbb{R}$, the ∇^{R}-holonomy group $H^{\mathrm{R}}|_{(x_0,s_0)} \subset \mathrm{SO}(T|_{(x_0,s_0)}(M \oplus \mathbb{R}))$ acts irreducibly and non-transitively on $T|_{(x_0,s_0)}(M \oplus \mathbb{R})$. Theorem 4.27 therefore implies that for all $(x_0, s_0) \in M \times \mathbb{R}$ the holonomy system $S_{(x_0,s_0)} := (T|_{(x_0,s_0)}(M \oplus \mathbb{R}), R^{\nabla^{\mathrm{R}}}|_{(x_0,s_t)}, H^{\mathrm{R}}|_{(x_0,s_0)})$ is symmetric. Notice that the fact that $S_{(x_0,s_0)}$ is a holonomy system in the first place follows from three facts: 1) $R^{\nabla^{\mathrm{R}}}|_{(x_0,s_t)}$ satisfies the equations (28)-(31), 2) $H^{\mathrm{R}}|_{(x_0,s_0)}$ is compact since it is a connected subgroup of $\mathrm{SO}(T|_{(x_0,s_0)}(M \times \mathbb{R}))$ acting irreducibly (see [16], Appendix 5) and 3) Ambrose-Singer theorem implies that $R^{\nabla^{\mathrm{R}}}|_{(x_0,s_t)}((X, r\partial_t), (Y, s\partial_t))$ always belongs to the Lie-algebra of $H^{\mathrm{R}}|_{(x_0,s_0)}$. Moreover, we have explicitly

$$R^{\nabla^{\mathrm{R}}}((X, r\partial_t), (Y, s\partial_t))(Z, u\partial_t) = R^{\nabla^{\mathsf{Rol}}}(X, Y)(Z, u) = (R(X, Y)Z - B(X, Y)Z, 0),$$

where $B(X,Y)Z := g(Y,Z)X - g(X,Z)Y$. Notice that $R^{\nabla^{\mathrm{R}}}$ cannot vanish identically on $M\times\mathbb{R}$, since in that case $H^{\mathrm{R}}|_{(x_0,s_0)}$ would be trivial by Ambrose-Singer Theorem, which contradicts the irreducibility of its action.

Consider the open set $O := \{(x,s) \in M\times\mathbb{R} \mid R^{\nabla^{\mathrm{R}}} \neq 0\}$. We claim that O is actually empty, which leads us to the sought contradiction. Indeed, suppose $(x_0,s_0),(x,s)\in O$ and choose some path $(\gamma,\tau):[0,1]\to M$ from (x,s) to (x_0,s_0). Then if $R_0^{\nabla^{\mathrm{R}}}$ denotes the parallel translate $(P^{\nabla^{\mathrm{R}}})_0^1(\gamma,\tau)R^{\nabla^{\mathrm{R}}}|_{(x,s)}$, then $R_0^{\nabla^{\mathrm{R}}}$ is a non-zero curvature tensor in $T|_{(x_0,s_0)}(M\times\mathbb{R})$. The Ambrose-Singer theorem implies that $(T|_{(x_0,s_0)}(M\times\mathbb{R}), R_0^{\nabla^{\mathrm{R}}}, H^{\mathrm{R}}|_{(x_0,s_0)})$ and $(T|_{(x_0,s_0)}(M\times\mathbb{R}), R^{\nabla^{\mathrm{R}}}|_{(x_0,s_0)}, H^{\mathrm{R}}|_{(x_0,s_0)})$ are both holonomy systems. Also, Theorem 4.27 implies that the holonomy system $(T|_{(x_0,s_0)}(M\times\mathbb{R}), R_0^{\nabla^{\mathrm{R}}}, H^{\mathrm{R}}|_{(x_0,s_0)})$ is a symmetric. Therefore, if one writes $V=T|_{(x_0,s_0)}(M\times\mathbb{R})$, $G=H^{\mathrm{R}}|_{(x_0,s_0)}$, $\mathcal{R}=R^{\nabla^{\mathrm{R}}}|_{(x_0,s_0)}\neq 0$, $\mathcal{R}'=R_0^{\nabla^{\mathrm{R}}}\neq 0$, Proposition 4.27 shows that there exists a unique $c\neq 0$ such that $(P^{\nabla^{\mathrm{R}}})_0^1(\gamma,\tau)R^{\nabla^{\mathrm{R}}}|_{(x,s)} = cR^{\nabla^{\mathrm{R}}}|_{(x_0,s_0)}$.

Let E be the exponential mapping of ∇^{R} starting at (x_0,s_0) and choose $\mathcal{U}\subset T|_{(x_0,s_0)}(M\times\mathbb{R})$ small enough such that this exponential mapping is a diffeomorphism of $\mathcal{U}$ onto an open subset $U\ni(x_0,s_0)$ of $M\times\mathbb{R}$ which is contained in the open set O. Then by the above formula, for every $(x,s)\in U$ one has a unique $f(x,s)\neq 0$ such that

$$f(x,s)(P^{\nabla^{\mathrm{R}}})_0^1\big(t\mapsto E(tE^{-1}(x,s))\big)R^{\nabla^{\mathrm{R}}}|_{(x_0,s_0)} = R^{\nabla^{\mathrm{R}}}|_{(x,s)}.$$

Clearly $(x,s)\mapsto f(x,s)$ is smooth. Let Γ be a ∇^{R}-geodesic through (x_0,s_0) Then since for t small, $f(\Gamma(t))R^{\nabla^{\mathrm{R}}}|_{(x_0,s_0)} = (P^{\nabla^{\mathrm{R}}})_t^0(\Gamma)R^{\nabla^{\mathrm{R}}}|_{\Gamma(t)}$, we get

$$\dot\Gamma(0)(f)R^{\nabla^{\mathrm{R}}}|_{(x_0,s_0)} = \nabla_{\dot\Gamma(0)}R^{\nabla^{\mathrm{R}}}. \tag{32}$$

If $\mathrm{pr}_2 : M\times\mathbb{R}\to\mathbb{R}$ is the projection onto the second factor, one sees from the explicit expression of $R^{\nabla^{\mathrm{R}}}$ that $(\mathrm{pr}_2)_*\big(R^{\nabla^{\mathrm{R}}}((Y,s\partial_t),(Z,u\partial_t))(W,v\partial_t)\big)=0$, for every $(Y,s\partial_t),(Z,u\partial_t),(W,v\partial_t)\in T|_{(x_0,s_0)}(M\times\mathbb{R})$. Thus (32) shows that $(\mathrm{pr}_2)_*\big(\nabla_{\dot\Gamma(0)}R^{\nabla^{\mathrm{R}}}((Y,s\partial_t),(Z,u\partial_t))(W,v\partial_t)\big)=0$.

Write $(X, r\partial_t|_{s_0}) = \dot\Gamma(0)$ and take $Y,Z,W\in\mathrm{VF}(M)$, $s,u,v\in C^\infty(M)$. Then one has

$$\begin{aligned}
&\nabla^{\mathrm{R}}_{\dot\Gamma(0)}\big(R^{\nabla^{\mathrm{R}}}((Y,s\partial_t),(Z,u\partial_t))(W,v\partial_t)\big) = (\nabla^{\mathrm{R}}_{\dot\Gamma(0)}R^{\nabla^{\mathrm{R}}})((Y,s\partial_t),(Z,u\partial_t))(W,v\partial_t)\big)\\
&+R^{\nabla^{\mathrm{R}}}(\nabla^{\mathrm{R}}_{\dot\Gamma(0)}(Y,s\partial_t),(Z,u\partial_t))(W,v\partial_t) + R^{\nabla^{\mathrm{R}}}((Y,s\partial_t),\nabla^{\mathrm{R}}_{\dot\Gamma(0)}(Z,u\partial_t))(W,v\partial_t)\\
&+R^{\nabla^{\mathrm{R}}}((Y,s\partial_t),(Z,u\partial_t))\nabla^{\mathrm{R}}_{\dot\Gamma(0)}(W,v\partial_t),
\end{aligned}$$

and hence, $(\mathrm{pr}_2)_*\big(\nabla^{\mathrm{R}}_{\dot\Gamma(0)}\big(R^{\nabla^{\mathrm{R}}}((Y,s\partial_t),(Z,u\partial_t))(W,v\partial_t)\big)\big)=0$. Moreover, one also has

$$\begin{aligned}
&(\mathrm{pr}_2)_*\nabla^{\mathrm{R}}_{\dot\Gamma(0)}\big(R^{\nabla^{\mathrm{R}}}((Y,s\partial_t),(Z,u\partial_t))(W,v\partial_t)\big)\\
=&(\mathrm{pr}_2)_*\nabla^{\mathrm{R}}_{(X,r\partial_t|_{s_0})}(R(Y,Z)W-B(Y,Z)W,0) = -g(X,R(Y,Z)W-B(Y,Z)W).
\end{aligned}$$

Hence $g(X,R|_{x_0}(Y,Z)W - B|_{x_0}(Y,Z)W)=0$ and since Y,Z,W,Γ, and thus X, were arbitrary, we deduce from this that $R|_{x_0}=B|_{x_0}$. But this then implies that $R^{\nabla^{\mathrm{R}}}|_{(x_0,s_0)}=0$, which is in contradiction with the definition of the set O containing (x_0,s_0). □

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C A Characterization of Isometries Between Riemannian Manifolds by using Development along Geodesic Triangles

(published in Archivum Mathematicum, tomus 48 (2012), Number 3)

A Characterization of Isometries Between Riemannian Manifolds by using Development along Geodesic Triangles

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February 28, 2012

Abstract

In this paper we characterize the existence of Riemannian covering maps from a complete simply connected Riemannian manifold (M, g) onto a complete Riemannian manifold $(\hat{M}, \hat{g})$ in terms of developing geodesic triangles of M onto $\hat{M}$. More precisely, we show that if $A_0 : T|_{x_0}M \to T|_{\hat{x}_0}\hat{M}$ is some isometric map between the tangent spaces and if for any two geodesic triangles γ, ω of M based at x_0 the development through A_0 of the composite path $\gamma.\omega$ onto $\hat{M}$ results in a closed path based at $\hat{x}_0$, then there exists a Riemannian covering map $f : M \to \hat{M}$ whose differential at x_0 is precisely A_0. The converse of this result is also true.

Keywords. Cartan-Ambrose-Hicks theorem, Development, Linear and affine connections, Rolling of manifolds.

MSC. 53B05, 53C05, 53B21.

1 Introduction

Consider two Riemannian manifolds (M, g) and $(\hat{M}, \hat{g})$ of the same dimension an suppose that one is given an isometry A_0 between given tangent spaces $T|_{x_0}M$ and $T|_{\hat{x}_0}\hat{M}$ of M and $\hat{M}$, respectively. Given a piecewise smooth path $\gamma : [0, 1] \to M$ starting from x_0, one *develops* this curve onto the tangent space $T|_{x_0}M$ to obtain a curve $\Gamma : [0, 1] \to T|_{x_0}M$ such that $\Gamma(t) = \int_0^t P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s$ where $P_s^0(\gamma)$ is the parallel transport on M along γ from $\gamma(s)$ to $\gamma(0) = x_0$. Consider the curve $\hat{\Gamma} := A_0 \circ \Gamma$ on $T|_{\hat{x}_0}\hat{M}$ and let $\hat{\gamma} : [0, 1] \to \hat{M}$ be the unique curve (if it exists) on $\hat{M}$, called the *anti-development of* $\hat{\Gamma}$, starting at $\hat{x}_0$ such that $\hat{\Gamma}(t) = \int_0^t P_s^0(\hat{\gamma})\dot{\hat{\gamma}}(s)\mathrm{d}s$ where $P_s^0(\hat{\gamma})$ is a parallel transport on $\hat{M}$ along $\hat{\gamma}$ from $\hat{\gamma}(s)$ to $\hat{\gamma}(0) = \hat{x}_0$. We say that $\hat{\gamma}$ is the development of γ onto $\hat{M}$ through A_0.

It happens, as it is easy to verify, that if (M, g) and $(\hat{M}, \hat{g})$ are isometric through an isomorphism $f : M \to \hat{M}$ whose differential at x_0 is A_0, that $\hat{\gamma} = f \circ \gamma$. Thus, in particular, if γ is a loop based at x_0, then $\hat{\gamma}$ will be a loop based at $\hat{x}_0$.

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This paper addresses the converse of this result: For a given A_0 as above, suppose that for every loop γ based at x_0 its development $\hat{\gamma}$ onto $\hat{M}$ through A_0 is a loop (necessarily based at $\hat{x}_0$), then does there exist an isomorphism $f : M \to \hat{M}$ whose differential at x_0 is A_0 ? Under the technical assumptions that (M, g) and $(\hat{M}, \hat{g})$ are complete and simply connected, we are able to answer affirmatively to this question. Indeed, instead of an arbitrary piecewise smooth loop γ based at x_0, it is enough to consider loops γ that are composites of two geodesic triangles based at x_0. Also, the assumptions of simply connectedness can be relaxed; see the main theorem 3.1 and its corollary 3.3.

This result is related to, and was originally inspired by, the so-called rolling model of Riemannian manifolds (cf. [1, 4, 5, 6, 7, 11, 13]). Consider two complete, oriented and simply-connected Riemannian manifolds (M, g), $(\hat{M}, \hat{g})$ of the same dimension and suppose A_0 is an oriented isometry from $T|_{x_0}M$ onto $T|_{\hat{x}_0}\hat{M}$, called an initial *relative orientation* of M and $\hat{M}$ at the initial *contact points* x_0 and $\hat{x}_0$. Let $\gamma : [0, 1] \to M$ be a piecewise smooth path on M such that $\gamma(0) = x_0$. Put M and $\hat{M}$ in contact at the points x_0 and $\hat{x}_0$, respectively, (here it might be useful to think of M and $\hat{M}$ as submanifolds of some $\mathbb{R}^N$ and $g, \hat{g}$ being the metrics induced by the Euclidean metric of $\mathbb{R}^N$) and identify the tangent spaces at these points by using A_0. Then let M *roll* against $\hat{M}$ along γ so that the motion contains no instantaneous *spinning* nor *slipping*. The set of contact points on $\hat{M}$ that are produced by this rolling motion form a piecewise smooth curve $\hat{\gamma}(t)$ i.e. at instant $t \in [0, 1]$ the contact point $\hat{\gamma}(t)$ of $\hat{M}$ corresponds to that of $\gamma(t)$ of M. In fact, the model explicitly tells that $P^0_t(\hat{\gamma})\dot{\hat{\gamma}}(t) = A_0 P^0_t(\gamma)\dot{\gamma}(t)$ for all $t \in [0, 1]$ i.e. $\hat{\gamma}$ is nothing more than the development of γ on $\hat{M}$ through A_0 as defined just above. Therefore, to detect if M and $\hat{M}$ are isomorphic, through some isomorphism $f : M \to \hat{M}$ with $f_*|_{x_0} = A_0$, it is enough to make M roll against $\hat{M}$ along loops of M based at x_0, identifying initially $T|_{x_0}M$ to $T|_{\hat{x}_0}\hat{M}$ through A_0, and to observe whether or not the paths so traced on $\hat{M}$ by the rolling motion are loops based at $\hat{x}_0$. Indeed, as mentioned above, it is even enough to consider the rolling along loops γ that are composites of two geodesic triangles based at x_0. This is a way of interpreting the main result, Theorem 3.1, of this paper in terms of a mechanical model and "physical experiments".

The outline of the paper is as follows. Section 2 introduces basic concepts, notations and results. The next section 3 contains the statement of the main theorem 3.1 of the paper along with its immediate corollaries. The proof of the main theorem is found in section 4. Actually, there we first prove a technical result (Proposition 4.1) in a more general context of affine manifolds and use it to prove the main theorem. Section 5 relates Theorem 3.1 to the well known Cartan-Ambrose-Hicks theorem ([2, 3, 10, 12]). We give in Theorem 5.2 a total of 8 different characterizations for the existence of a Riemannian covering map between two Riemannian manifolds, one of which is the Cartan-Ambrose-Hicks theorem and one is the main theorem of the paper. Finally, section 6 contains an application to the main theorem related to the affine holonomy group of a Riemannian manifold ([8]).

2 Notations and Basic Results

All the manifolds that appear are assumed to be smooth, second countable and Hausdorff (cf. [9, 12]). If $M, \hat{M}$ are manifolds and $x \in M$, $\hat{x} \in \hat{M}$, we write $T^*|_x M \otimes T|_{\hat{x}}\hat{M}$ for the set of all $\mathbb{R}$-linear maps $T|_x M \to T|_{\hat{x}}\hat{M}$. We define $T^*M \otimes T\hat{M} := \bigcup_{(x,\hat{x}) \in M \times \hat{M}} T|_x M \otimes T|_{\hat{x}}\hat{M}$ for the set of all linear maps between different tangent spaces.

If M is a manifold and $x \in M$, write $\Omega_x(M)$ for the set of all piecewise smooth loops

$\gamma : [0,1] \to M$ based at x i.e. $\gamma(0) = \gamma(1) = x$. If $\gamma : [a,b] \to M$ and $\omega : [c,d] \to M$ are paths such that $\gamma(b) = \omega(c)$ we define the *composite path* as

$$\omega \sqcup \gamma : [a, b+d-c] \to M; \quad \omega \sqcup \gamma(t) = \begin{cases} \gamma(t), & \text{if } t \in [a,b] \\ \omega(t-b+c), & \text{if } t \in [b, b+d-c]. \end{cases}$$

If $a = c = 0$ and $b = d = 1$, i.e. $\gamma, \omega : [0,1] \to M$, then one defines the composite path $\omega.\gamma$ as

$$\omega.\gamma : [0,1] \to M; \quad \omega.\gamma := \begin{cases} \gamma(2t), & t \in [0, \frac{1}{2}] \\ \omega(2t-1), & t \in [\frac{1}{2}, 1], \end{cases}.$$

i.e. $\omega.\gamma = (t \mapsto \omega(2t))|_{[0,1/2]} \sqcup (t \mapsto \gamma(2t))|_{[0,1/2]}$. The inverse path $\gamma^{-1} : [a,b] \to M$ of $\gamma : [a,b] \to M$ is defined as $\gamma^{-1}(t) = \gamma(b+a-t)$. Observe that if $\gamma : [a,b] \to M$,$\omega : [c,d] \to M$ and $\Gamma : [A,B] \to M$ are three path such that $\gamma(b) = \omega(c)$ and $\omega(d) = \Gamma(A)$, then $(\Gamma \sqcup \omega) \sqcup \gamma = \Gamma \sqcup (\omega \sqcup \gamma)$. However, if $\gamma, \omega, \Gamma : [0,1] \to M$ and $\gamma(1) = \omega(0)$, $\omega(1) = \Gamma(0)$, then $\Gamma.(\omega.\gamma) \neq (\Gamma.\omega).\gamma$. This lack of associativity for '.'-operation will not be a handicap for us, as will be explained in Remark 2.9 below, and usually we prefer working with "normalized" paths whose domain of definition is $[0,1]$.

A manifold M equipped with a linear connection ∇ is called an *affine manifold* (M, ∇). If $\gamma : [a,b] \to M$ is a piecewise smooth path and (M, ∇) is an affine manifold, we write $(P^\nabla)^t_s(\gamma)$, where $t, s \in [a,b]$, for the ∇-parallel transport from $\gamma(s)$ to $\gamma(t)$. Since the connection to be used is usually clear from the context, we write simply $P^t_s(\gamma)$ for $(P^\nabla)^t_s(\gamma)$. The exponential map of (M, ∇) at x is written as $\exp^\nabla_x$ and (M, ∇) is said to be *geodesically accesible* from $x \in M$ if $\exp^\nabla_x$ is surjective onto M. If $\exp^\nabla_x$ is defined on the whole tangent space $T|_x M$ for all $x \in M$, then (M, ∇) is said to be *geodesically complete*. The curvature (resp. torsion) tensor on (M, ∇) is denoted by R^∇ (resp. T^∇). If $(\hat{M}, \hat{\nabla})$ is another affine manifold, then a smooth map $f : M \to \hat{M}$ is called *affine*, if for any piecewise smooth path $\gamma : [a,b] \to M$, one has

$$f_*|_{\gamma(b)} \circ (P^\nabla)^b_a(\gamma) = (P^{\hat{\nabla}})^b_a(f \circ \gamma) \circ f_*|_{\gamma(a)}.$$

A manifold M equipped with a positive definite (i.e. Riemannian) metric g is called a Riemannian manifold (M, g). If (M, g) and $(\hat{M}, \hat{g})$ are Riemannian manifolds and if $A \in T^*|_x M \otimes T|_{\hat{x}} \hat{M}$ is such that $\hat{g}(AX, AY) = g(X, Y)$ for all $X, Y \in T|_x M$, we say that A is an *infinitesimal isometry*.

Definition 2.1 Let (M, ∇) be an affine manifold and $k \geq 1$.

(i) A path $\gamma : [a,b] \to M$ is called a *k-times broken geodesic*, if there are geodesics $\gamma_0, \dots, \gamma_k$ such that γ_i ends where γ_{i+1} starts from and $\gamma = \gamma_k \sqcup \gamma_{k-1} \sqcup \dots \sqcup \gamma_1 \sqcup \gamma_0$.

We use $\angle_x(M, \nabla)$ to denote the set of 1-times broken geodesics defined on $[0,1]$ and starting from $x \in M$.

(ii) A loop $\gamma \in \Omega_x(M)$ based at x is said to be a *geodesic k-polygon* based at x if it is a $(k-1)$-times broken geodesic.

Geodesic 3-polygons (resp. 4-polygons) based at x are called geodesic triangles (resp. quadrilaterals) based x and they constitute a set denoted by $\triangle_x(M, \nabla)$ (resp. $\square_x(M, \nabla)$). We also define

$$\triangle^2_{x_0}(M, \nabla) := \{\gamma.\omega \mid \gamma, \omega \in \triangle_{x_0}(M, \nabla)\}.$$

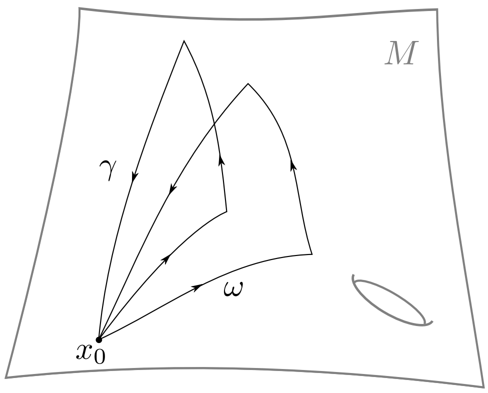

Figure 1: A typical element $\gamma.\omega$ of $\triangle^2_{x_0}(M,\nabla)$.

Remark 2.2 Notice that a path $\gamma : [a,b] \to M$ is a k-times broken geodesic if and only if there is a partition $\{t_0,\dots,t_{k+1}\}$ of $[a,b]$ such that $\gamma|_{[t_i,t_{i+1}]}$ is a ∇-geodesic for $i=0,\dots,k$.

Definition 2.3 Let (M,∇) and $(\hat{M},\hat{\nabla})$ be affine manifolds and let $A \in T^*|_x M \otimes T|_{\hat{x}}\hat{M}$. We define $\mathcal{T}^{(\nabla,\hat{\nabla})}_A : \bigwedge^2 T|_x M \to T|_{\hat{x}}\hat{M}$ and $\mathcal{R}^{(\nabla,\hat{\nabla})}_A : \bigwedge^2 T|_x M \to T^*|_x M \otimes T|_{\hat{x}}\hat{M}$ called the *relative torsion* and *relative curvature* of (M,∇), $(\hat{M},\hat{\nabla})$ at A by

$$\begin{aligned}\mathcal{T}^{(\nabla,\hat{\nabla})}_A(X,Y) :=& AT^{\nabla}(X,Y) - T^{\hat{\nabla}}(AX,AY)\\ \mathcal{R}^{(\nabla,\hat{\nabla})}_A(X,Y)Z :=& A(R^{\nabla}(X,Y)Z) - R^{\hat{\nabla}}(AX,AY)AZ,\end{aligned}$$

where $X,Y,Z \in T|_x M$. We will often write simply $\mathcal{T}_A$ and $\mathcal{R}_A$ for $\mathcal{T}^{(\nabla,\hat{\nabla})}_A$ and $\mathcal{R}^{(\nabla,\hat{\nabla})}_A$, respectively, when $\nabla,\hat{\nabla}$ are clear from the context.

Definition 2.4 Let (M,∇) be an affine manifold. For a piecewise smooth $\gamma : [a,b] \to M$ we define a piecewise smooth $\Lambda^{\nabla}_{\gamma(a)}(\gamma) : [a,b] \to T|_{\gamma(a)}M$ by

$$\Lambda^{\nabla}_{\gamma(a)}(\gamma)(t) = \int_a^t (P^{\nabla})^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s, \quad t \in [a,b]$$

We call $\Lambda^{\nabla}_{\gamma(a)}(\gamma)$ the *development of* γ on $T|_{\gamma(a)}M$ with respect to the connection ∇.

In the Riemannian setting, one can characterize the completeness in terms of the development map.

Theorem 2.5 ([8], Theorem IV.4.1) A Riemannian manifold (M,g), with Levi-Civita connection ∇, is complete if and only if for every $x \in M$ and every piecewise smooth curve $\Gamma : [a,b] \to T|_x M$, $\Gamma(a) = 0$, there exists a (necessarily unique) piecewise smooth curve $\gamma : [a,b] \to M$ such that $\gamma(a) = x$ and $\Lambda^{\nabla}_x(\gamma) = \Gamma$.

Definition 2.6 Given (M,∇), $(\hat{M},\hat{\nabla})$, $A_0 \in T^*|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ and a piecewise smooth $\gamma : [a,b] \to M$ such that $\gamma(a) = x_0$.

(i) We define

$$\Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\gamma)(t) := (\Lambda^{\hat{\nabla}}_{\hat{x}_0})^{-1}(A_0 \circ \Lambda^{\nabla}_{x_0}(\gamma))(t),$$

for all $t \in [a,b]$ where defined. If $c \in [a,b]$ is such that $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(c)$ exists, we call $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)|_{[a,c]}$ the *development* of γ onto $\hat{M}$ through A_0 with respect to $(\nabla, \hat{\nabla})$. We will usually write simply $\Lambda_{A_0}(\gamma)$ for $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)$ when there is no risk of confusion.

(ii) If $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(t)$ is defined, we define the *relative parallel transport of* A_0 along γ to be the linear map

$$(\mathcal{P}^{(\nabla,\hat{\nabla})})_a^t(\gamma)A_0 : T|_{\gamma(t)}M \to T|_{\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(t)}\hat{M};$$
$$(\mathcal{P}^{(\nabla,\hat{\nabla})})_a^t(\gamma)A_0 := (P^{\hat{\nabla}})_a^t\big(\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)\big) \circ A_0 \circ (P^{\nabla})_t^a(\gamma).$$

As before, one writes briefly $\mathcal{P}_a^t(\gamma)A_0$ for $(\mathcal{P}^{(\nabla,\hat{\nabla})})_a^t(\gamma)A_0$ when the connections in question are evident.

Remark 2.7 It is evident that $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(t)$ exists for all $t > a$ near a and if $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(t)$ exists for some $t > a$ then $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(t')$ exists for all $t' \in [a,t]$ as well. By Theorem 2.5, if $(\hat{M}, \hat{g})$ is a complete Riemannian manifold with Levi-Civita connection $\hat{\nabla}$, the development $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(t)$ is defined for every $t \in [a,b]$.

We record a lemma whose easy proof we omit.

Lemma 2.8 Let (M, ∇) and $(\hat{M}, \hat{\nabla})$ be affine manifold, $A_0 \in T^*|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ and $\gamma : [a,b] \to M$ a piecewise smooth path with $\gamma(a) = x_0$.

(i) If $\omega : [c,d] \to M$ is a piecewise smooth path such that $\gamma(b) = \omega(c)$, then

$$\Lambda_{A_0}(\omega \sqcup \gamma) = \Lambda_{\mathcal{P}_a^b(\gamma)A_0}(\omega) \sqcup \Lambda_{A_0}(\gamma)$$
$$\mathcal{P}_a^t(\omega \sqcup \gamma)A_0 = \begin{cases} \mathcal{P}_a^t(\gamma)A_0, & \text{if } t \in [a,b] \\ \mathcal{P}_c^{t-b+c}(\omega)\mathcal{P}_a^b(\gamma)A_0, & \text{if } t \in [b, b+d-c]. \end{cases}$$

Moreover,

$$\Lambda_{A_0}(\gamma^{-1} \sqcup \gamma)(2b-a) = x_0, \quad \mathcal{P}_a^{2b-a}(\gamma^{-1} \sqcup \gamma)A_0 = A_0.$$

(ii) If $\gamma : [a,b] \to M$ is a k-times broken geodesic on (M, ∇) and if $\Lambda_{A_0}(\gamma)(t)$ exists for all $t \in [a,b]$ then $\Lambda_{A_0}(\gamma)$ is a k-times broken geodesic on $(\hat{M}, \hat{\nabla})$. In particular, if $\gamma_u(t) := \exp_{x_0}^{\nabla}(tu)$, $\hat{\gamma}_{A_0u}(t) := \exp_{\hat{x}_0}^{\hat{\nabla}}(tA_0u)$, then $\Lambda_{A_0}(\gamma_u) = \hat{\gamma}_{A_0u}$.

(iii) Let $\hat{\gamma} : [a,b] \to \hat{M}$ be a piecewise smooth curve such that $\hat{\gamma}(a) = \hat{x}_0$. Then $\hat{\gamma} = \Lambda_{A_0}(\gamma)$ if and only if

$$P_t^a(\hat{\gamma})\dot{\hat{\gamma}}(t) = A_0 P_t^a(\gamma)\dot{\gamma}(t), \quad t \in [a,b].$$

(iv) If $X(\cdot)$ is a vector field along $\gamma : [a,b] \to M$, then for all $t \in [a,b]$ such that $\hat{\gamma}(t) := \Lambda_{A_0}(\gamma)(t)$ is defined, one has

$$\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\big((\mathcal{P}_a^t(\gamma)A_0)X(t)\big) = (\mathcal{P}_a^t(\gamma)A_0)\nabla_{\dot{\gamma}(t)}X(t).$$

(v) Suppose $\phi : [\alpha, \beta] \to [a, b]$ is smooth and $\dot{\phi}(t) \neq 0$ $\forall t \in [\alpha, \beta]$. Then the following hold for all $t \in [\alpha, \beta]$ such that left or right hand side is defined:

$$\Lambda_{A_0}(\gamma)(\phi(t)) = \Lambda_{A_0}(\gamma \circ \phi)(t)$$
$$\mathcal{P}_a^{\phi(t)}(\gamma)A_0 = \mathcal{P}_\alpha^t(\gamma \circ \phi)A_0.$$

Remark 2.9 Suppose $\gamma, \omega, \Gamma : [0, 1] \to M$ are such that $\gamma(1) = \omega(0)$, $\omega(1) = \Gamma(0)$. As remarked earlier, $\Gamma.(\omega.\gamma) \neq (\Gamma.\omega).\gamma$. Lemma 2.8, however, implies that

$$\Lambda_{A_0}((\Gamma.\omega).\gamma)(1) = \Lambda_{A_0}(\Gamma.(\omega.\gamma))(1)$$
$$\mathcal{P}_0^1((\Gamma.\omega).\gamma)A_0 = \mathcal{P}_0^1(\Gamma.(\omega.\gamma))A_0$$

Indeed, the latter (which implies the former) follows by computing

$$\begin{aligned}\mathcal{P}_0^1((\Gamma.\omega).\gamma)A_0 =& \mathcal{P}_0^1(\Gamma.\omega)\mathcal{P}_0^1(\gamma)A_0 = \mathcal{P}_0^1(\Gamma)\mathcal{P}_0^1(\omega)\mathcal{P}_0^1(\gamma)A_0 = \mathcal{P}_0^1(\Gamma)\mathcal{P}_0^1(\omega.\gamma)A_0 \\ =& \mathcal{P}_0^1(\Gamma.(\omega.\gamma))A_0.\end{aligned}$$

We recall next the Cartan-Ambrose-Hicks theorem (C-A-H Theorem for short) in the context of Riemannian manifolds of equal dimension.

Theorem 2.10 (C-A-H) ([2, 3, 10, 12]) Let (M, g) and $(\hat{M}, \hat{g})$ be complete Riemannian manifolds of the same dimension, $\dim M = \dim \hat{M}$, and let $A_0 \in T^*|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ be an infinitesimal isometry. Then there exists a complete Riemannian manifold (N, h), $z_0 \in N$ and Riemannian covering maps $F : (N, h) \to (M, g)$, $G : (N, h) \to (\hat{M}, \hat{g})$ such that $G_*|_{z_0} = A_0 \circ F_*|_{z_0}$ if and only if

$$\mathcal{R}^{(\nabla, \hat{\nabla})}_{\mathcal{P}_0^1(\gamma)A_0} = 0, \quad \forall \gamma \in \angle_{x_0}(M, \nabla), \tag{1}$$

where $\nabla, \hat{\nabla}$ are the Levi-Civita connections of (M, g) and $(\hat{M}, \hat{g})$.

3 Main Result

We begin this section with the statement of the main theorem of the paper. The result will then be followed by two corollaries and some remarks. The proof of the theorem is postponed to section 4.

Theorem 3.1 Suppose (M, g), $(\hat{M}, \hat{g})$ are complete Riemannian manifolds of the same dimension, $\dim M = \dim \hat{M}$, M simply connected and let $A_0 \in T|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ be an infinitesimal isometry. Then there exists a Riemannian covering map $f : M \to \hat{M}$ with $f_*|_{x_0} = A_0$ if and only if

$$\Lambda^{(\nabla, \hat{\nabla})}_{A_0}\big(\triangle^2_{x_0}(M, \nabla)\big) \subset \Omega_{\hat{x}_0}(\hat{M}), \tag{2}$$

where $\nabla, \hat{\nabla}$ are the Levi-Civita connections of (M, g), $(\hat{M}, \hat{g})$, respectively.

Remark 3.2 Notice that by Lemma 2.8 (ii) the condition (2) is equivalent to

$$\Lambda^{(\nabla, \hat{\nabla})}_{A_0}\big(\triangle^2_{x_0}(M, \nabla)\big) \subset \triangle^2_{\hat{x}_0}(\hat{M}, \hat{\nabla})$$

and that it is implied by the condition

$$\Lambda^{(\nabla, \hat{\nabla})}_{A_0}\big(\Omega_{x_0}(M)\big) \subset \Omega_{\hat{x}_0}(\hat{M}). \tag{3}$$

If one wishes not to assume M to be simply connected in Theorem 3.1, then the result can be modified as follows:

Corollary 3.3 Suppose (M, g), $(\hat{M}, \hat{g})$ are complete Riemannian manifolds of the same dimension, $\dim M = \dim \hat{M}$ and let $A_0 \in T|_{x_0} M \otimes T|_{\hat{x}_0} \hat{M}$ be and infinitesimal isometry. The condition (2) holds if and only if there exists a complete simply connected Riemannian manifold (N, h), Riemannian covering maps $F : N \to M$, $G : N \to \hat{M}$ and a $z_0 \in N$ such that $G_*|_{z_0} = A_0 \circ F_*|_{z_0}$ and

$$\{\Gamma(1) \mid \Gamma : [0,1] \to N \text{ continuous},\ \Gamma(0) = z_0,\ F \circ \Gamma \in \triangle^2_{x_0}(M, \nabla)\} \subset G^{-1}(\hat{x}_0), \qquad (4)$$

Proof. Sufficiency. Let (N, h) and the maps F, G be given as stated and suppose (4) is true. For a $\gamma \in \triangle^2_{x_0}(M, \nabla)$, let Γ be the unique path in N such that $\gamma = F \circ \Gamma$ and $\Gamma(0) = z_0$. It follows that $G \circ \Gamma = \Lambda_{A_0}(\gamma)$ and since $\Gamma(1) \in G^{-1}(\hat{x}_0)$, we have $\Lambda_{A_0}(\gamma)(1) = G(\Gamma(1)) = \hat{x}_0$ i.e. (2) holds.

Necessity. Let $F : N \to M$ be the universal covering of M and lift the metric g onto N, which we denote by h. As is well known, (N, h) is complete. Fix a point $z_0 \in F^{-1}(x_0)$ and write D for the Levi-Civita connection of (N, h). Let $B_0 := A_0 \circ F_*|_{z_0} \in T^*|_{z_0} N \otimes T|_{\hat{x}_0} \hat{M}$ which is an infinitesimal isometry and notice that if $\Gamma : [0,1] \to N$ is a piecewise smooth path starting from z_0, then $\Lambda_{B_0}(\Gamma) = \Lambda_{A_0}(F \circ \Gamma)$. In particular, if $\Gamma \in \triangle^2_{z_0}(N, D)$, then $F \circ \Gamma \in \triangle^2_{x_0}(M, \nabla)$ and hence $\Lambda_{B_0}(\Gamma) \in \Omega_{\hat{x}_0}(\hat{M})$ by assumption (2).

Thus Theorem 3.1 implies the existence of a Riemannian covering map $G : N \to \hat{M}$ such that $G_*|_{z_0} = B_0 = A_0 \circ F_*|_{z_0}$. To prove (4), let $\Gamma : [0,1] \to N$ be such that $\Gamma(0) = z_0$ and $F \circ \Gamma \in \triangle^2_{x_0}(M, \nabla)$. Then $G \circ \Gamma = \Lambda_{B_0}(\Gamma) = \Lambda_{A_0}(F \circ \Gamma) \in \Omega_{\hat{x}_0}(\hat{M})$ so in particular, $G(\Gamma(1)) = \Lambda_{A_0}(F \circ \Gamma)(1) = \hat{x}_0$ i.e. $\Gamma(1) \in G^{-1}(\hat{x}_0)$. The proof is complete. $\square$

Remark 3.4 If the the previous corollary one replaces the condition (2) by (3), then (4) can be replaced by the condition $F^{-1}(x_0) \subset G^{-1}(\hat{x}_0)$. This is clear from the proof of the corollary.

The above theorem has an easy corollary.

Corollary 3.5 Let (M, g), $(\hat{M}, \hat{g})$ be complete Riemannian manifolds of the same dimension, $\dim M = \dim \hat{M}$. Given an infinitesimal isometry $A_0 \in T|_{x_0} M \otimes T|_{\hat{x}_0} \hat{M}$ and $x_1 \in M$, $\hat{x}_1 \in \hat{M}$, then there exists a Riemannian covering map $f : M \to \hat{M}$ with $f_*|_{x_0} = A_0$ and $f(x_1) = \hat{x}_1$ if and only if

$$\forall \text{ 6-broken geodesic } \gamma : [0,1] \to M \text{ s.t. } \gamma(0) = x_0,\ \gamma(1) = x_1 \quad \Longrightarrow \quad \Lambda^{(\nabla, \hat{\nabla})}_{A_0}(\gamma)(1) = \hat{x}_1.$$

Proof. Necessity. Suppose we are given a Riemannian covering map $f : M \to \hat{M}$ with $f_*|_{x_0} = A_0$ and $f(x_1) = \hat{x}_1$. Then if γ is a 6-broken geodesic with $\gamma(0) = x_0$, $\gamma(1) = x_1$, it follows that $\Lambda_{A_0}(\gamma) = f \circ \gamma$ and hence $\Lambda_{A_0}(\gamma)(1) = f(\gamma(1)) = f(x_1) = \hat{x}_1$.

Sufficiency. Let $\Gamma : [0,1] \to M$ be any geodesic from x_0 to x_1 (such a geodesic exists since (M, g) is complete) and define $A_1 := \mathcal{P}^1_0(\Gamma) A_0$. Taking any $\gamma, \omega \in \triangle_{x_1}(M, \nabla)$, we see that (see Lemma 2.8)

$$\Lambda_{A_1}(\gamma.\omega)(1) = \Lambda_{\mathcal{P}^1_0(\Gamma) A_0}(\gamma.\omega)(1) = \Lambda_{A_0}((\gamma.\omega).\Gamma)(1) = \hat{x}_1,$$

where the last equality follows from the fact that $(\gamma.\omega).\Gamma$ is a 6-broken geodesic that starts from x_0 and ends to x_1. Thus $\Lambda_{A_1}(\triangle^2_{x_1}(M, \nabla)) \subset \Omega_{\hat{x}_1}(\hat{M})$ and Theorem 3.1 implies that there is a covering map $f : M \to \hat{M}$ such that $f_*|_{x_1} = A_1$. In particular, $f(x_1) = \hat{x}_1$. Moreover,

$$f_*|_{\Gamma^{-1}(t)} = \mathcal{P}_0^t(\Gamma^{-1})A_1 = \mathcal{P}_0^{1-t}A_0,$$

which implies that $f_*|_{x_0} = f_*|_{\Gamma^{-1}(1)} = \mathcal{P}_0^0 A_0 = A_0$. The proof is finished. $\square$

Remark 3.6 The condition of the corollary means that 6-times broken geodesics of M with end points x_0 and x_1 map by the development $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}$ to 6-times broken geodesics of $\hat{M}$ with end points $\hat{x}_0$ and $\hat{x}_1$.

Also observe that this condition is implied by the following stronger one:

$$\forall\, \gamma : [0,1] \to M, \text{ piecewise smooth s.t. } \gamma(0) = x_0,\ \gamma(1) = x_1 \quad \Longrightarrow \quad \Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(1) = \hat{x}_1.$$

4 Proof of the Main Result

The proof of Theorem 3.1 (see p.15) makes use of the following key proposition which we state and prove in a more general setting of affine manifolds.

Proposition 4.1 Suppose that (M, ∇), $(\hat{M}, \hat{\nabla})$ are affine manifolds (possibly of different dimensions) and let $A_0 \in T|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ be given. Let $U \subset T|_{x_0}M$, $\hat{U} \subset T|_{\hat{x}_0}\hat{M}$ be the domains of definitions of $\exp_{x_0}^{\nabla}$, $\exp_{\hat{x}_0}^{\hat{\nabla}}$, respectively, and write $\gamma_u(t) = \exp_{x_0}^{\nabla}(tu)$, $\hat{\gamma}_{\hat{u}}(t) = \exp_{\hat{x}_0}^{\hat{\nabla}}(t\hat{u})$ for $u \in U$, $\hat{u} \in \hat{U}$. Then if

$$\Lambda_{A_0}^{(\nabla,\hat{\nabla})}\big(\triangle_{x_0}(M, \nabla)\big) \subset \triangle_{\hat{x}_0}(\hat{M}, \hat{\nabla}) \tag{5}$$

$$\mathcal{T}^{(\nabla,\hat{\nabla})}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), \cdot) = 0, \quad \forall u \in U \cap A_0^{-1}(\hat{U}), \tag{6}$$

hold, one has for all $u \in U \cap A_0^{-1}(\hat{U})$ that

$$(P^{\hat{\nabla}})_1^0(\hat{\gamma}_{A_0u}) \circ (\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{A_0u} \circ A_0 = A_0 \circ (P^{\nabla})_1^0(\gamma_u) \circ (\exp_{x_0}^{\nabla})_*|_u \tag{7}$$

$$\mathcal{R}^{(\nabla,\hat{\nabla})}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), \cdot)\dot{\gamma}_u(1) = 0. \tag{8}$$

Remark 4.2 Since $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)$ might not be defined on whole interval $[0,1]$ for every $\gamma \in \triangle_{x_0}(M, \nabla)$, except if e.g. $(\hat{M}, \hat{\nabla})$ is geodesically complete, we understand the assumption (5) to mean that if $\gamma \in \triangle_{x_0}(M, \nabla)$ and if $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)$ is defined on $[0,1]$, then $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma) \in \triangle_{\hat{x}_0}(\hat{M}, \hat{\nabla})$.

Proof. We will not make the assumption Eq. (6) until later on. Notice also that U and $\hat{U}$ are star-shaped around the origin of $T|_{x_0}M$ and hence so is $U \cap A_0^{-1}(\hat{U})$. In the proof we write $\gamma_X(t) = \exp_x^{\nabla}(tX)$ and $\hat{\gamma}_{\hat{X}}(t) = \exp_{\hat{x}}^{\hat{\nabla}}(t\hat{X})$ whenever $x \in M$, $\hat{x} \in \hat{M}$ and $X \in T|_xM$, $\hat{X} \in T|_{\hat{x}}\hat{M}$ and $t \in \mathbb{R}$ are such that these are defined. If they are defined for all $t \in [0,1]$, we assume, by default, that the domains of definitions of γ_X and $\hat{\gamma}_{\hat{X}}$ are the interval $[0,1]$.

Given $u \in U \cap A_0^{-1}(\hat{U})$ and $v \in T|_{x_0}M$ we define a vector field along γ_u by

$$Y_{u,v}(t) := \frac{\partial}{\partial s}\Big|_0 \exp_{x_0}^{\nabla}(t(u + sv)) = t(\exp_{x_0}^{\nabla})_*|_{tu}(v)$$

i.e. $Y_{u,v}$ is the unique Jacobi field along γ_u such that $Y_{u,v}(0) = 0$, $\nabla_{\dot{\gamma}_u(t)} Y_{u,v}|_{t=0} = v$. Moreover, we write $C^T_{x_0}$ for the set of tangent conjugate points in $T|_{x_0}M$ of $\exp^{\nabla}_{x_0}$ i.e.

$$\begin{aligned} C^T_{x_0} =& \{u \in U \mid \exists v \in T_{x_0}M,\ v \neq 0 \text{ s.t. } Y_{u,v}(1) = 0\} \\ =& \{u \in U \mid \text{rank } (\exp^{\nabla}_{x_0})_*|_u < \dim M\}. \end{aligned}$$

Fix, for now, $u \in U \cap A_0^{-1}(\hat{U})$, $v \in T|_{x_0}M$, and assume that $u \notin C^T_{x_0}$. Let V_u be an open neighbourhood of u in $T|_{x_0}M$ such that $\exp^{\nabla}_{x_0}|_{V_u}$ is a diffeomorphism and $V_u \subset A_0^{-1}(\hat{U})$. Define $\omega_{u,w} \in \triangle_{x_0}(M, \nabla)$, for all $w \in T|_{x_0}M$ near enough to the origin such that $\exists \gamma_{Y_{u,w}(1)}(t) \in \exp^{\nabla}_{x_0}(V_u)$ for all $t \in [0, 1]$, by

$$\omega_{u,w} := \gamma^{-1}_{Z_{u,w}}.(\gamma_{Y_{u,w}(1)}.\gamma_u),$$

where $Z_{u,w} := (\exp^{\nabla}_{x_0}|_{V_u})^{-1}(\gamma_{Y_{u,w}(1)}(1))$. For such a w we also define

$$\hat{\omega}_{u,w}(t) := \Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\omega_{u,w})(t), \quad t \in [0, 1],$$

which exists if w is near enough to the origin in $T|_{x_0}M$. Notice that, by assumption, $\hat{\omega}_{u,w} \in \triangle_{\hat{x}_0}(\hat{M}, \hat{\nabla})$.

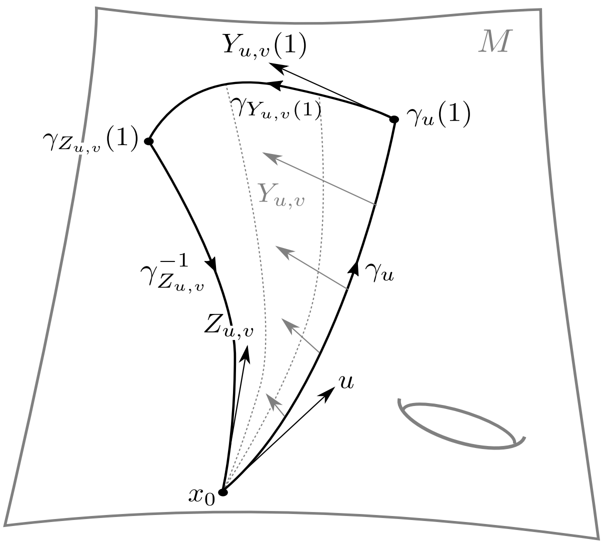

Figure 2: Construction of the geodesic triangle $\omega_{u,v} = \gamma^{-1}_{Z_{u,v}}.(\gamma_{Y_{u,v}(1)}.\gamma_u)$.

In particular, if $s \in \mathbb{R}$ is near zero, $\omega_{u,sv}$ and $\hat{\omega}_{u,sv}$ are defined and

$$\hat{\omega}_{u,sv}(t) = \Lambda^{(\nabla,\hat{\nabla})}_{A_0}(\omega_{u,sv})(t).$$

It follows that (see Lemma 2.8 case (ii)) for every s near zero, the curve $t \mapsto \hat{\omega}^{-1}_{u,sv}(t/2)$, $t \in [0, 1]$, is a geodesic and $\hat{\omega}^{-1}_{u,0}(t/2) = \hat{\gamma}_{A_0 u}(t)$ for $t \in [0, 1]$. Therefore a vector field along $\hat{\gamma}_{A_0 u}$ defined by

$$\hat{Y}_{u,v}(t) := \frac{\partial}{\partial s}\Big|_0 \hat{\omega}^{-1}_{u,sv}(t/2), \quad t \in [0, 1]$$

is a Jacobi field. Since $\hat{\omega}_{u,sv}^{-1}(0) = \hat{\omega}_{u,sv}(1) = \hat{x}_0$, we have that $\hat{Y}_{u,v}(0) = 0$ which implies that there is a unique $\hat{v}(u,v) \in T|_{\hat{x}_0}\hat{M}$ such that

$$\hat{Y}_{u,v}(t) = \frac{\partial}{\partial s}\big|_0 \exp_{\hat{x}_0}^{\hat{\nabla}}(t(A_0u + s\hat{v}(u,v))) = t(\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}(\hat{v}(u,v)), \quad t \in [0,1], \tag{9}$$

i.e. $\hat{v}(u,v) = \hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\hat{Y}_{u,v}(t)|_{t=0}$. Notice that $\hat{Y}_{u,v}(t)$, and hence also $\hat{v}(u,v)$, is well defined for all $(u,v) \in (U \cap A_0^{-1}(\hat{U}) \setminus C_{x_0}^T) \times T|_{x_0}M$ and $t \in [0,1]$ and it is clear that $\hat{v}$ is a smooth map.

We will now state and prove three lemmas and come back to the proof of the proposition after them.

Lemma 4.3 Under the above assumptions, one has

$$\hat{\nabla}_{\dot{\hat{\gamma}}_u(t)}\hat{Y}_{u,v}|_{t=1} = (\mathcal{P}_0^1(\gamma_u)A_0)\nabla_{\dot{\gamma}_u(t)}Y_{u,v}|_{t=1} - \mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1)). \tag{10}$$

for all $(u,v) \in (U \cap A_0^{-1}(\hat{U}) \setminus C_{x_0}^T) \times T|_{x_0}M$.

Proof. In the proof we always assume that $s \in \mathbb{R}$ that appears is near zero. Then we may assume that $\omega_{u,sv}$, $\hat{\omega}_{u,sv}$ and $\hat{Y}_{u,sv}$ are defined.

Writing $\partial_t := \frac{\partial}{\partial t}\exp_{x_0}^{\nabla}(t(u+sv))$, $\partial_s := \frac{\partial}{\partial s}\exp_{x_0}^{\nabla}(t(u+sv))$ $\hat{\partial}_t := \frac{\partial}{\partial t}\hat{\omega}_{u,sv}^{-1}(t/2)$, $\hat{\partial}_s := \frac{\partial}{\partial s}\hat{\omega}_{u,sv}^{-1}(t/2)$, we have (here $\frac{\partial}{\partial t}|_{1-}$ means the left hand side derivative at $t=1$)

$$\begin{aligned}
&\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\hat{Y}_{u,v}|_{t=1} - T^{\hat{\nabla}}(\hat{\partial}_t, \hat{\partial}_s)|_{(t,s)=(1,0)}\\
=&\hat{\nabla}_{\hat{\partial}_t}\frac{\partial}{\partial s}\Big|_0 \hat{\omega}_{u,sv}^{-1}(t/2)\Big|_{t=1-} - T^{\hat{\nabla}}(\hat{\partial}_t, \hat{\partial}_s)|_{(t,s)=(1,0)}\\
=&\hat{\nabla}_{\hat{\partial}_s}\frac{\partial}{\partial t}\Big|_{1-}\hat{\omega}_{u,sv}^{-1}(t/2)\Big|_{s=0} = -\hat{\nabla}_{\hat{\partial}_s}\frac{\partial}{\partial t}\Big|_{1-}\hat{\omega}_{u,sv}(1-t/2)\Big|_{s=0}\\
=&-\hat{\nabla}_{\hat{\partial}_s}\Big(\mathcal{P}_0^{1/2}(\omega_{u,sv})A_0\frac{\partial}{\partial t}\Big|_{1-}\omega_{u,sv}(1-t/2)\Big)\Big|_{s=0}\\
=&\hat{\nabla}_{\hat{\partial}_s}\Big(\mathcal{P}_0^1(\gamma_{Y_{u,sv}(1)}.\gamma_u)A_0\frac{\partial}{\partial t}\big|_1 \exp_{x_0}^{\nabla}(tZ_{u,sv})\Big)\Big|_{s=0}
\end{aligned}$$

where at the second to last equality we used the fact that $\dot{\hat{\omega}}_{u,sv}(t) = (\mathcal{P}_0^t(\omega_{u,sv})A_0)\dot{\omega}_{u,sv}(t)$, $t \in [0,1]$ (using one-sided derivatives at break points); see Lemma 2.8 case (iii). Notice that $Y_{u,sv}(t) = sY_{u,v}(t)$ and so $\gamma_{Y_{u,sv}(1)}(t) = \gamma_{sY_{u,v}(1)}(t) = \gamma_{Y_{u,v}(1)}(ts)$, which leads us to conclude that if $s > 0$,

$$\mathcal{P}_0^1(\gamma_{Y_{u,sv}(1)}.\gamma_u)A_0 = \mathcal{P}_0^1(t \mapsto \gamma_{Y_{u,v}}(ts))\mathcal{P}_0^1(\gamma_u)A_0 = \mathcal{P}_0^s(\gamma_{Y_{u,v}(1)})\mathcal{P}_0^1(\gamma_u)A_0.$$

Therefore

$$\begin{aligned}
&\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\hat{Y}_{u,v}|_{t=1} - T^{\hat{\nabla}}(\hat{\partial}_t, \hat{\partial}_s)|_{(t,s)=(1,0)}\\
=&\hat{\nabla}_{\hat{\partial}_s}\Big((\mathcal{P}_0^s(\gamma_{Y_{u,v}(1)})\mathcal{P}_0^1(\gamma_u)A_0)\frac{\partial}{\partial t}\big|_1 \exp_{x_0}^{\nabla}(tZ_{u,sv})\Big)\Big|_{s=0+}\\
=&\big(\mathcal{P}_0^1(\gamma_u)A_0\big)\nabla_{\partial_s}\big(\frac{\partial}{\partial t}\big|_1 \exp_{x_0}^{\nabla}(tZ_{u,sv})\big)\big|_{s=0}\\
=&\big(\mathcal{P}_0^1(\gamma_u)A_0\big)\Big(\nabla_{\dot{\gamma}_u(t)}\big(\frac{\partial}{\partial s}\big|_0 \exp_{x_0}^{\nabla}(tZ_{u,sv})\big)\big|_{t=1} + T(\partial_s, \partial_t)|_{(t,s)=(1,0)}\Big),
\end{aligned}$$

where at the second to last equality we used Lemma 2.8 case (iv) (notice that $s \mapsto \frac{\partial}{\partial t}\big|_1 \exp^{\nabla}_{x_0}(tZ_{u,sv})$ is a vector field along $s \mapsto \gamma_{Y_{u,v}(1)}(s) = \gamma_{Y_{u,sv}(1)}(1)$) and at the last equality we noticed that $Z_{u,0} = u$, so $\frac{\partial}{\partial t} \exp^{\nabla}_{x_0}(tZ_{u,sv})\big|_{s=0} = \dot{\gamma}_u(t) = \partial_t$. At this moment we make the following observations:

$$
\begin{aligned}
\hat{\partial}_t|_{(t,s)=(1,0)} =& (\mathcal{P}_0^1(\gamma_u)A_0)\partial_t|_{(t,s)=(1,0)} \\
\partial_s|_{(t,s)=(1,0)} =& Y_{u,v}(1), \\
\hat{\partial}_s|_{(t,s)=(1,0)} =& \hat{Y}_{u,v}(1) = \frac{\partial}{\partial s}\big|_0 \hat{\omega}^{-1}_{u,sv}(1/2) = \frac{\partial}{\partial s}\big|_0 \hat{\omega}_{u,sv}(1/2) = \frac{\partial}{\partial s}\big|_0 \Lambda_{A_0}(\gamma_{Y_{u,sv}(1)}.\gamma_u)(1) \\
=& \frac{\partial}{\partial s}\big|_0 \Lambda_{\mathcal{P}_0^1(\gamma_u)A_0}(\gamma_{Y_{u,sv}(1)})(1) = \frac{\partial}{\partial s}\big|_0 \hat{\gamma}_{(\mathcal{P}_0^1(\gamma_u)A_0)Y_{u,sv}(1)}(1) \\
=& \frac{\partial}{\partial s}\big|_0 \hat{\gamma}_{(\mathcal{P}_0^1(\gamma_u)A_0)Y_{u,v}(1)}(s) = (\mathcal{P}_0^1(\gamma_u)A_0)Y_{u,v}(1).
\end{aligned}
$$

These allow us to write the above equation into the form

$$
\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0 u}(t)} \hat{Y}_{u,v}|_{t=1} + \mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1)) = \big(\mathcal{P}_0^1(\gamma_u)A_0\big)\nabla_{\dot{\gamma}_u(t)}\big(\frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(tZ_{u,sv})\big)\big|_{t=1}.
$$

Therefore, it remains to show that $\frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(tZ_{u,sv}) = Y_{u,v}(t)$, $\forall t \in [0,1]$. Indeed, $J(t) := \frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(tZ_{u,sv})$ is a Jacobi field along γ_u and it satisfies the boundary conditions $J(0) = 0 = Y_{u,v}(0)$ and

$$
J(1) = \frac{\partial}{\partial s}\big|_0 \exp^{\nabla}_{x_0}(Z_{u,sv}) = \frac{\partial}{\partial s}\big|_0 \gamma_{Y_{u,sv}(1)}(1) = \frac{\partial}{\partial s}\big|_0 \gamma_{Y_{u,v}(1)}(s) = Y_{u,v}(1).
$$

Since $u \notin C^T_{x_0}$, it follows that $J = Y_{u,v}$ and thus the proof is finished. □

From the last proof, we record for later use the following fact:

$$
\hat{Y}_{u,v}(1) = (\mathcal{P}_0^1(\gamma_u)A_0)Y_{u,v}(1), \tag{11}
$$

for all $(u,v) \in (U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}) \times T|_{x_0}M$.

Lemma 4.4 Under the above assumptions, for all $(u,v) \in (U \cap A_0^{-1}(\hat{U}) \setminus C^T_{x_0}) \times T|_{x_0}M$ the following holds:

$$
(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0 u}(\partial_1 \hat{v}(u,v)(u)) = \mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1)).
$$

Hence in particular,

$$
\mathcal{T}_{A_0} = 0.
$$

Proof. By assumption, $u \in U \cap A_0^{-1}(\hat{U}) \setminus C^T_{x_0}$ and so for all t near 1, one has $tu \in U \cap A_0^{-1}(\hat{U}) \setminus C^T_{x_0}$. In the proof of the first claim, we assume always that t is near enough to 1 so that this is the case.

Since $Y_{tu,v}(1) = \frac{1}{t} Y_{u,v}(t)$, Eq. (11) implies that

$$
t\hat{Y}_{tu,v}(1) = t(\mathcal{P}_0^1(\gamma_{tu})A_0)Y_{tu,v}(1) = (\mathcal{P}_0^t(\gamma_u)A_0)Y_{u,v}(t).
$$

Writing $\partial_t := \dot{\gamma}_u(t)$, $\hat{\partial}_t := \dot{\hat{\gamma}}_{A_0u}(t)$ to simplify the notation, we have

$$\begin{aligned}
&\hat{Y}_{u,v}(1) + \hat{\nabla}_{\hat{\partial}_t}\hat{Y}_{tu,v}(1)|_{t=1} = \hat{\nabla}_{\hat{\partial}_t}(t\hat{Y}_{tu,v}(1))|_{t=1} = \hat{\nabla}_{\hat{\partial}_t}\big((\mathcal{P}_0^t(\gamma_u)A_0)Y_{u,v}(t)\big)|_{t=1} \\
=&(\mathcal{P}_0^t(\gamma_u)A_0)\nabla_{\partial_t}Y_{u,v}(t)|_{t=1} = \hat{\nabla}_{\hat{\partial}_t}\hat{Y}_{u,v}(t)|_{t=1} + \mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1)),
\end{aligned}$$

where at the third equality we used Lemma 2.8 case (iv) and at the fourth equality we used (10). But

$$\begin{aligned}
&\hat{\nabla}_{\hat{\partial}_t}\hat{Y}_{u,v}(t)|_{t=1} = \hat{\nabla}_{\hat{\partial}_t}\big(t(\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}(\hat{v}(u,v))\big)\big|_{t=1} \\
=&(\exp_{\hat{x}_0})_*|_{A_0u}(\hat{v}(u,v)) + \hat{\nabla}_{\hat{\partial}_t}\big((\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}(\hat{v}(u,v))\big)\big|_{t=1} \\
=&\hat{Y}_{u,v}(1) + \hat{\nabla}_{\hat{\partial}_t}\big((\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}(\hat{v}(u,v))\big)\big|_{t=1},
\end{aligned}$$

while

$$\hat{\nabla}_{\hat{\partial}_t}\hat{Y}_{tu,v}(1)|_{t=1} = \hat{\nabla}_{\hat{\partial}_t}\big((\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}(\hat{v}(tu,v))\big)\big|_{t=1},$$

so combining these three formulas, one gets

$$\begin{aligned}
\mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1)) =&\hat{\nabla}_{\hat{\partial}_t}\big((\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}(\hat{v}(tu,v))\big)\big|_{t=1} - \hat{\nabla}_{\hat{\partial}_t}\big((\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}(\hat{v}(u,v))\big)\big|_{t=1} \\
=&\hat{\nabla}_{\hat{\partial}_t}\big((\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}\big(\hat{v}(tu,v) - \hat{v}(u,v)\big)\big)\big|_{t=1}.
\end{aligned}$$

Writing $\partial_1\hat{v}(u,v)(X)$ for the differential of $\hat{v}$ at (u,v) with respect to v in the direction X, we have

$$\hat{v}(tu,v) - \hat{v}(u,v) = \int_1^t \frac{\partial}{\partial s}\hat{v}(su,v)\mathrm{d}s = \int_1^t \partial_1\hat{v}(su,v)(u)\mathrm{d}s.$$

Notice that $(\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u} \in T^*(T|_{\hat{x}_0}\hat{M}) \otimes T\hat{M}$ for $t \in [0,1]$, so if we write $\hat{D}$ for the vector bundle connection on $T^*(T|_{\hat{x}_0}\hat{M}) \otimes T\hat{M} \to T|_{\hat{x}_0}\hat{M} \times \hat{M}$ naturally induced by the canonical connection on vector space $T|_{\hat{x}_0}\hat{M}$ and $\hat{\nabla}$, we get finally

$$\begin{aligned}
&\mathcal{T}_{\mathcal{P}_0^1(\gamma_u)A_0}(\dot{\gamma}_u(1), Y_{u,v}(1)) = \hat{\nabla}_{\hat{\partial}_t}\Big((\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}\Big(\int_1^t \partial_1\hat{v}(su,v)(u)\mathrm{d}s\Big)\Big)\Big|_{t=1} \\
=&\Big(\hat{D}_{\frac{\mathrm{d}}{\mathrm{d}t}(tA_0u, \gamma_{A_0u}(t))}(\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tA_0u}\Big)\Big|_{t=1}\int_1^1 \partial_1\hat{v}(su,v)(u)\mathrm{d}s \\
&+ (\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{A_0u}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\int_1^t \partial_1\hat{v}(su,v)(u)\mathrm{d}s \\
=&(\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{A_0u}(\partial_1\hat{v}(u,v)(u)),
\end{aligned}$$

which proves the first part of the lemma.

It remains to prove that $\mathcal{T}_{A_0} = 0$. Indeed, by what was just proved, we have that for all $u, v \in T|_{x_0}M$ and for all t small,

$$(\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tu}(\partial_1\hat{v}(tu,v)(tu)) = \mathcal{T}_{\mathcal{P}(\gamma_{tu},A_0)(1)}(\dot{\gamma}_{tu}(1), Y_{tu,v}(1))$$

holds i.e., because $Y_{tu,v}(1) = \frac{1}{t}Y_{u,v}(t)$,

$$t(\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{tu}(\partial_1\hat{v}(tu,v)(u)) = \mathcal{T}_{\mathcal{P}(\gamma_u,A_0)(t)}(\dot{\gamma}_u(t), \frac{1}{t}Y_{u,v}(t)).$$

But as $t \to 0$, one has $\frac{1}{t}Y_{u,v}(t) \to \nabla_{\dot{\gamma}_u(t)}Y_{u,v}|_{t=0} = v$, $\partial_1\hat{v}(tu,v)(u) \to \partial_1\hat{v}(0,v)(u)$ and $\mathcal{P}(\gamma_u, A_0)(t) \to A_0$, so in the limit one gets $0 = \mathcal{T}_{A_0}(u,v)$. Since $u, v \in T|_{x_0}M$ were arbitrary, the result follows. $\square$

From now on we will make all the assumption in the statement of Proposition 4.1 i.e. we also include the torsion condition Eq. (6).

Lemma 4.5 Under the above assumptions and for all $(u, v) \in (U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}) \times T|_{x_0} M$ and $t \in [0, 1]$, one has

$$\hat{Y}_{u,v}(t) = \frac{\partial}{\partial s}\big|_0 \exp^{\hat{\nabla}}_{\hat{x}_0}(tA_0(u + sv)) \tag{12}$$

$$\hat{Y}_{u,v}(t) = (\mathcal{P}^t_0(\gamma_u) A_0) Y_{u,v}(t). \tag{13}$$

Proof. Write $\hat{C}^T_{\hat{x}_0}$ for the tangent conjugate set of $(\hat{M}, \hat{\nabla})$ at $\hat{x}_0$ defined in the same way as $C^T_{x_0}$. By Lemma 4.4 and condition (6), one has $(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0 u}(\partial_1 \hat{v}(u, v)(u)) = 0$ for all $(u, v) \in (U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}) \times T|_{x_0} M$. Given such a (u, v), if $A_0 u \notin \hat{C}^T_{\hat{x}_0}$, then $\partial_1 \hat{v}(u, v)(u) = 0$. Otherwise $A_0 u \in \hat{C}^T_{\hat{x}_0}$, but then $\exists \epsilon > 0$ such that $tu \in U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}$ and $tA_0 u \notin \hat{C}^T_{\hat{x}_0}$ for all $t \in]1 - \epsilon, 1 + \epsilon[\backslash \{1\}$, hence $\partial_1 \hat{v}(tu, v)(tu) = 0$. Letting $t \to 1$ then implies that $\partial_1 \hat{v}(u, v)(u) = 0$. Therefore we have shown that

$$\partial_1 \hat{v}(u, v)(u) = 0, \quad \forall (u, v) \in (U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}) \times T|_{x_0} M.$$

Now fix $u \in U \cap A_0^{-1}(\hat{U}) \backslash C^T_{x_0}$ and $v \in T|_{x_0} M$. Notice that the set $S := \{t \in [0, 1] \mid tu \in C^T_{x_0}\}$ is finite or empty. If $S \neq \emptyset$, we write $S = \{t_i\}_{i=1,\dots,N}$ where $0 < t_i < t_{i+1} < 1$ for all i and we set $t_{N+1} := 1$. In the case where S is empty, we set $t_1 := 1$, $N := 0$.

Write $t_0 := 0$ and notice that for $t, \tau \in]t_i, t_{i+1}[$ we have

$$\hat{v}(tu, v) = \hat{v}(\tau u, v) + \int_\tau^t \frac{1}{s} \underbrace{\partial_1 \hat{v}(su, v)(su)}_{=0} \,\mathrm{d}s = \hat{v}(\tau u, v),$$

i.e. the value of $t \mapsto \hat{v}(tu, v)$ is constant on each interval $]t_i, t_{i+1}[$, $i = 0, \dots, N$. Let $\hat{v}_i(u, v)$ be the constant value of $\hat{v}(tu, v)$ for $t \in]t_i, t_{i+1}[$.

Define $\hat{J}_{u,v}(t)$, $t \in [0, 1] \backslash S$, by

$$\hat{J}_{u,v}(t) := t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0 u}(\hat{v}(tu, v)), \quad \text{if } t \in]t_i, t_{i+1}[.$$

Then $\hat{J}_{u,v}$ is a Jacobi field on each interval $]t_i, t_{i+1}[$ since for $t \in]t_i, t_{i+1}[$.

$$\hat{J}_{u,v}(t) = t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0 u}(\hat{v}_i(u, v)) = \frac{\partial}{\partial s}\big|_0 \exp^{\hat{\nabla}}_{\hat{x}_0}\big(t(A_0 u + s\hat{v}_i(u, v))\big).$$

But we observe that

$$\hat{J}_{u,v}(t) = t\hat{Y}_{tu,v}(1) = (\mathcal{P}^t_0(\gamma_u) A_0) Y_{u,v}(t), \quad \forall t \in [0, 1] \backslash S,$$

hence because $t \mapsto (\mathcal{P}^t_0(\gamma_u) A_0) Y_{u,v}(t)$ is smooth and S is finite, we see that $\hat{J}_{u,v}(t)$ uniquely extends to a Jacobi field along $\hat{\gamma}_{A_0 u}$ defined on the whole interval $[0, 1]$. We still denote this Jacobi field by $\hat{J}_{u,v}(t)$ and notice that since

$$\hat{J}_{u,v}(t) = t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0 u}(\hat{v}_0(u, v))$$

holds for $t \in]0, t_1[$, it holds for all $t \in [0, 1]$.

To identify $\hat{J}_{u,v}(t)$ once and for all, it remains to compute the value of $\hat{v}_0(u,v)$. We have

$$\begin{aligned}\hat{v}(0,v) =&(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_0(\hat{v}(0,v)) = \hat{Y}_{0,v}(1) = (\mathcal{P}^1_0(\gamma_0)A_0)Y_{0,v}(1)\\ =&A_0 Y_{0,v}(1) = A_0(\exp^{\nabla}_{x_0})_*|_0(v) = A_0 v\end{aligned}$$

and thus $\hat{v}_0(u,v) = \lim_{t\to 0+}\hat{v}(tu,v) = \hat{v}(0,v) = A_0 v$. We have thus shown the following:

$$(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0u}(\hat{v}(u,v)) = \hat{Y}_{u,v}(1) = \hat{J}_{u,v}(1) = (\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0u}(A_0 v).$$

We will prove that $\hat{v}(u,v) = A_0 v$. Indeed, if $A_0 u \notin \hat{C}^T_{\hat{x}_0}$, then the above equation readily implies that $\hat{v}(u,v) = A_0 v$. On the other hand, if $A_0 u \in \hat{C}^T_{\hat{x}_0}$, then for all $t \neq 1$ near 1, one has $tA_0 u \notin \hat{C}^T_{\hat{x}_0}$ and $(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(\hat{v}(tu,v) - A_0 v) = 0$, which implies that $\hat{v}(tu,v) = A_0 v$ and finally $\hat{v}(u,v) = A_0 v$ by passing to the limit $t \to 1$.

Since $\hat{v}(u,v) = A_0 v$, the claimed eq. (12) follows from (9). To prove (13), notice that

$$(\mathcal{P}^t_0(\gamma_u)A_0)Y_{u,v}(t) = t\hat{Y}_{tu,v}(1) = t\frac{\partial}{\partial s}\big|_0 \exp^{\hat{\nabla}}_{\hat{x}_0}(A_0(tu+sv)) = t(\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{tA_0u}(A_0 v) = \hat{Y}_{u,v}(t).$$

This concludes the proof.

□

We are now ready to finish the proof of the proposition. Let $u \in U \cap A_0^{-1}(\hat{U})\backslash C^T_{x_0}$. Because $Y_{u,v}(1) = (\exp^{\nabla}_{x_0})_*|_u(v)$ by definition and since $\hat{Y}_{u,v}(1) = (\exp^{\hat{\nabla}}_{\hat{x}_0})_*|_{A_0u}(A_0 v)$, by (12), the formula (7) is an immediate consequence of (13) and Definition 2.6. Since $C^T_{x_0}$ has no interior points in $T|_{x_0}M$, it follows that (7) holds for all $u \in U \cap A_0^{-1}(\hat{U})$.

It remains to prove the formula (8). Let $(u,v) \in U \cap A_0^{-1}(\hat{U})\backslash C^T_{x_0} \times T|_{x_0}M$. Taking twice the covariant derivative w.r.t. $\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}$ of both sides of the equation (13), recalling that $Y_{u,v}$, $\hat{Y}_{u,v}$ are Jacobi fields and using Lemma 2.8 case (iv), we get

$$\begin{aligned}&R^{\hat{\nabla}}(\dot{\hat{\gamma}}_{A_0u}(t), \hat{Y}_{u,v}(t))\dot{\hat{\gamma}}_{A_0u}(t) + \hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\big(T^{\hat{\nabla}}(\dot{\hat{\gamma}}_{A_0u}(t), \hat{Y}_{u,v}(t))\big)\\ =&\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(\cdot)}\hat{Y}_{u,v}(\cdot) = (\mathcal{P}^t_0(\gamma_u)A_0)\nabla_{\dot{\gamma}_u(t)}\nabla_{\dot{\gamma}_u(\cdot)}Y_{u,v}(\cdot)\\ =&(\mathcal{P}^t_0(\gamma_u)A_0)\big(R^{\nabla}(\dot{\gamma}_u(t), Y_{u,v}(t))\dot{\gamma}_u(t) + \nabla_{\dot{\gamma}_u(t)}\big(T^{\nabla}(\dot{\gamma}_u(t), Y_{u,v}(t))\big)\big).\end{aligned}$$

Using the last two equations above, the fact that $\dot{\hat{\gamma}}_{A_0u}(t) = \mathcal{P}(\gamma_u, A_0)(t)\dot{\gamma}_u(t)$ and Definition 2.3 we get that, for all $(u,v) \in U \cap A_0^{-1}(\hat{U})\backslash C^T_{x_0} \times T|_{x_0}M$,

$$\mathcal{R}_{\mathcal{P}^t_0(\gamma_u)A_0}(\dot{\gamma}_u(t), Y_{u,v}(t))\dot{\gamma}_u(t) = -\hat{\nabla}_{\dot{\hat{\gamma}}_{A_0u}(t)}\mathcal{T}_{\mathcal{P}^{\cdot}_0(\gamma_u)A_0}(\dot{\gamma}_u(\cdot), Y_{u,v}(\cdot)) = 0,$$

since $\mathcal{T}_{\mathcal{P}^t_0(\gamma_u)A_0}(\dot{\gamma}_u(t), Y_{u,v}(t)) = \mathcal{T}_{\mathcal{P}^1_0(\gamma_{tu})A_0}(\dot{\gamma}_{tu}(1), Y_{tu,v}(1)) = 0$, $t \in [0,1]$, by assumtion (6).

Let then $u \in U \cap A_0^{-1}(U)$ and let $0 < t_1 < t_2 < \dots$ be the conjugate times along γ_u (i.e $\{t_1u, t_2u, \dots\} = \{tu \mid t \in [0,1]\} \cap C^T_{x_0}$). Suppose $X(t)$ is any vector field along γ_u. If $t \neq t_j$ for all j, then there is a $v(t) \in T|_{x_0}M$ such that $Y_{u,v(t)}(t) = X(t)$, and so

$$\mathcal{R}_{\mathcal{P}^t_0(\gamma_u)A_0}(\dot{\gamma}_u(t), X(t))\dot{\gamma}_u(t) = 0.$$

By continuity, this holds for all $t \in [0,1]$ and hence the result follows once we set $t = 1$.

□

Remark 4.6 Notice that (5) is equivalent to the condition

$$\Lambda_{A_0}^{(\nabla,\hat{\nabla})}\big(\triangle_{x_0}(M,\nabla)\big) \subset \Omega_{\hat{x}_0}(\hat{M}).$$

On the other hand, if (5) is replaced by a stronger condition

$$\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\Omega_{x_0}(M)) \subset \Omega_{\hat{x}_0}(\hat{M}),$$

then in the proof one can define $\omega_{u,v}$ to be $\gamma_{u+v}^{-1}.\big((s \mapsto \gamma_{u+sv}(1)).\gamma_u\big) \in \Omega_{x_0}(M)$. Then $\omega_{u,v}$ and hence $\hat{\omega}_{u,v}$, $\hat{Y}_{u,v}$ and finally $\hat{v}(u,v)$ are defined for all $(u,v) \in (U \cap A_0^{-1}(\hat{U})) \times T|_{x_0}M$. The proof goes through in the same way as above, all the lemmas 4.3-4.5 being true even with the set $U \cap A_0^{-1}(\hat{U}) \backslash C_{x_0}^T$ replaced with $U \cap A_0^{-1}(\hat{U})$ everywhere. Moreover, the proof becomes slightly easier since one does not need to pay attention to the tangent conjugate set $C_{x_0}^T$.

We will now proceed to the proof of Theorem 3.1.

Proof. (of Theorem 3.1) *Necessity.* If $f : M \to \hat{M}$ is a Riemannian covering with $f_*|_{x_0} = A_0$, then for $\gamma \in \triangle_{x_0}^2(M,\nabla)$, one has $\Lambda_{A_0}(\gamma) = f \circ \gamma$ and hence $\Lambda_{A_0}(\gamma)(1) = f(\gamma(1)) = f(x_0) = \hat{x}_0$. So $\Lambda_{A_0}(\gamma) \in \Omega_{\hat{x}_0}(\hat{M})$.

Sufficiency. The idea is to prove, using Proposition 4.1, that the condition of C-A-H Theorem 2.10 given by Eq. (1) holds, which then implies the claim.

Define

$$\mathcal{A} := \{\mathcal{P}_0^1(\gamma)A_0 \mid \gamma \in \triangle_{x_0}(M,\nabla)\}$$

and notice that assumption (5) implies that $\mathcal{A} \subset T^*|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ and it is clear that each $A \in \mathcal{A}$ is an infinitesimal isometry.

We claim that

$$P_1^0(\hat{\gamma}_{Au}) \circ (\exp_{\hat{x}_0}^{\hat{\nabla}})_*|_{Au} \circ A = A \circ P_1^0(\gamma_u) \circ (\exp_{x_0}^{\nabla})_*|_u, \quad \forall A \in \mathcal{A},\ u \in T|_{x_0}M. \tag{14}$$

Indeed, fix $A \in \mathcal{A}$ and let $\omega \in \triangle_{x_0}(M,\nabla)$ be arbitrary. Then there is an $\gamma \in \triangle_{x_0}(M,\nabla)$ such that $A = \mathcal{P}_0^1(\gamma)A_0$. But then $\omega.\gamma \in \triangle_{x_0}^2(M,\nabla)$ and hence by Lemma 2.8 (i) and the assumptions of the theorem,

$$\Lambda_A(\omega)(1) = \big(\Lambda_{\mathcal{P}_0^1(\gamma)A_0}(\omega).\Lambda_{A_0}(\gamma)\big)(1) = \Lambda_{A_0}(\omega.\gamma)(1) = \hat{x}_0$$

i.e. $\Lambda_A(\triangle_{x_0}(M,\nabla)) \subset \triangle_{\hat{x}_0}(\hat{M},\hat{\nabla})$. Thus the above claim follows from Proposition 4.1.

For a unit vector $u \in T|_{x_0}M$, let $\tau(u) \in]0,+\infty]$ be cut-time for the geodesic γ_u and set

$$U^T := \{tu \mid u \in T|_{x_0}M,\ \|u\|_g = 1,\ 0 \le t < \tau(u)\}, \quad U := \exp_{x_0}(U^T).$$

For every $A \in \mathcal{A}$ one defines a map

$$\phi_A : U \to \hat{M}; \quad \phi_A = \exp_{\hat{x}_0}^{\hat{\nabla}} \circ A \circ (\exp_{x_0}^{\nabla}|_{U^T})^{-1}.$$

We are to show that each ϕ_A is an isometry onto its open image. Indeed, if $x \in U$ and $X \in T|_xM$, let $u = (\exp_{x_0}|_{U^T})^{-1}(x)$ and use (14) to compute

$$\|(\phi_A)_*(X)\|_{\hat{g}} = \left\|(P_0^1(\hat{\gamma}_{Au}) \circ A \circ P_1^0(\gamma_u))X\right\|_{\hat{g}} = \left\|(A \circ P_1^0(\gamma_u))X\right\|_{\hat{g}} = \left\|P_1^0(\gamma_u)X\right\|_g = \|X\|_g.$$

Since $\dim M = \dim \hat{M}$, it follows that ϕ_A is a diffeomorphism onto its (open) image and is isometric. This settles the claim.

Knowing this, we may now show a property which then allows us (eventually) to call for the C-A-H Theorem 2.10: For all $A \in \mathcal{A}$ and all unit vectors $u \in T|_{x_0}M$,

$$\mathcal{R}_{\mathcal{P}_0^t(\gamma_u)A} = 0, \quad \forall t;\ 0 \le t \le \tau(u) \tag{15}$$

with the understanding that $0 \le t \le \tau(u)$ is replaced by $t \ge 0$ if $\tau(u) = +\infty$. To prove this, notice that since ϕ_A is an isometry onto its open image, one has

$$(\phi_A)_*\big(R^{\nabla}((X,Y)Z\big) = R^{\hat{\nabla}}(((\phi_A)_*X, (\phi_A)_*Y)((\phi_A)_*Z), \quad \forall x \in U,\ X,Y,Z \in T|_xM$$

i.e. $\mathcal{R}_{(\phi_A)_*|_x} = 0$ for all $x \in U$. But we know from (14) that if $0 \le t < \tau(u)$ (whence $tu \in U^T$), one has $(\phi_A)_*|_{\gamma_u(t)} = P_0^t(\gamma_{Au}) \circ A \circ P_t^0(\gamma_u)$, which equals to $\mathcal{P}_0^t(\gamma_u)A$. Hence $\mathcal{R}_{\mathcal{P}_0^t(\gamma_u)A} = 0$ if $0 \le t < \tau(u)$ and by continuity this also holds when $t = \tau(u)$ (if $\tau(u) < +\infty$) which establishes the claim.

We are now ready to finish the proof by appealing to C-A-H Theorem 2.10. Indeed, let $\omega \in \angle_{x_0}(M,\nabla)$. Since (M,g) is complete, there exists a unit vector $u \in T|_{x_0}M$ such that $\gamma_u : [0,\tau(u)] \to M$ is a minimal geodesic from x_0 to $\omega(1)$. Because then $\gamma_{\tau(u)u}^{-1}.\omega \in \triangle_{x_0}(M,\nabla)$, one has that $A := \mathcal{P}_0^1(\gamma_{\tau(u)u}^{-1}.\omega)A_0$ is in $\mathcal{A}$ and therefore $\mathcal{R}_{\mathcal{P}_0^{\tau(u)}(\gamma_u)A} = 0$ by (15). But by Lemma 2.8,

$$\mathcal{P}_0^{\tau(u)}(\gamma_u)A = \mathcal{P}_0^1(\gamma_{\tau(u)u})\mathcal{P}_0^1(\gamma_{\tau(u)u}^{-1}.\omega)A_0 = \mathcal{P}_0^1(\gamma_{\tau(u)u})\mathcal{P}_0^1(\gamma_{\tau(u)u}^{-1})\mathcal{P}_0^1(\omega)A_0 = \mathcal{P}_0^1(\omega)A_0,$$

which proves that $\mathcal{R}_{\mathcal{P}_0^1(\omega)A_0} = 0$ for all $\omega \in \angle_{x_0}(M,\nabla)$.

Therefore the condition (1) of C-A-H Theorem 2.10 is satisfied and hence there exists a complete Riemannian manifold (N,h), $z_0 \in N$ and Riemannian covering maps $F : N \to M$, $G : N \to \hat{M}$ such that $A_0 = G_*|_{z_0} \circ (F_*|_{z_0})^{-1}$. Since M is simply connected, $F : N \to M$ is a Riemannian isomorphism and setting $f := G \circ F^{-1}$ finishes the proof.

□

Remark 4.7 In the case where there are no cut-points on any geodesic of (M,g) emanating from x_0, then one may replace (2) in Theorem 3.1 by the condition

$$\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\triangle_{x_0}(M,\nabla)) \subset \Omega_{\hat{x}_0}(\hat{M}).$$

Indeed, in this case $\exp_{x_0}^{\nabla} : T|_{x_0}M \to M$ is a diffeomorphism and in the above proof $U^T = T|_{x_0}M$, $U = M$ and so $\phi_{A_0} : M \to \hat{M}$ is an isometry onto its open image. It follows from a standard result on Riemannian manifolds that $f := \phi_{A_0}$ is a covering map and obviously $f_*|_{x_0} = A_0$.

5 Different Formulations of the Cartan-Ambrose-Hicks Theorem

In this section, we will complement the C-A-H theorem 2.10 by giving eight equivalent characterizations for the existence of a Riemannian covering map $f : (M,g) \to (\hat{M},\hat{g})$ (under specific assumptions).

First we recall a well-known proposition and, for the sake of completeness, give its easy proof.

Proposition 5.1 Suppose (M,∇), $(\hat{M},\hat{\nabla})$ are affine manifolds such that M is simply connected and geodesically accessible from x_0 and that $(\hat{M},\hat{\nabla})$ is geodesically complete. Given $A_0 \in T|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$, there exists an affine map $f : M \to \hat{M}$ such that $f_*|_{x_0} = A_0$ if and only if

$$(\mathcal{P}^{(\nabla,\hat{\nabla})})_0^1(\gamma)A_0 = A_0, \quad \forall \gamma \in \square_{x_0}(M,\nabla). \tag{16}$$

Proof. Necessity. If $f : M \to \hat{M}$ is an affine map such that $f_*|_{x_0} = A_0$ and if $\gamma \in \square_{x_0}(M,\nabla)$, then $\Lambda_{A_0}(\gamma) = f \circ \gamma$ and hence $\Lambda_{A_0}(\gamma)(1) = f(\gamma(1)) = f(x_0) = \hat{x}_0$.

Sufficiency. In the proof we write $\gamma_u(t) = \exp_x^{\nabla}(tu)$, when $u \in T|_xM$. Let $x \in M$ be given. Let $\gamma, \omega \in \angle_{x_0}(M,\nabla)$ be such that $\gamma(1) = x$, $\omega(1) = x$, which exist since (M,∇) is geodesically accessible from x_0. Then $\omega^{-1}.\gamma \in \square_{x_0}(M,\nabla)$ and hence

$$\mathcal{P}_0^1(\omega^{-1}.\gamma)A_0 = A_0.$$

It follows that (see Lemma 2.8 and Remark 2.9)

$$\begin{aligned}\Lambda_{A_0}(\omega)(1) =& \Lambda_{\mathcal{P}_0^1(\omega^{-1}.\gamma)A_0}(\omega)(1) = \Lambda_{A_0}(\omega.(\omega^{-1}.\gamma))(1) = \Lambda_{A_0}((\omega.\omega^{-1}).\gamma)(1)\\ =& \Lambda_{\mathcal{P}_0^1(\gamma)A_0}(\omega.\omega^{-1})(1) = \Lambda_{A_0}(\gamma)(1).\end{aligned}$$

This shows that if for $x \in M$ one defines

$$f(x) := \{\Lambda_{A_0}(\gamma)(1) \mid \gamma \in \angle_{x_0}(M,\nabla),\ \gamma(1) = x\},$$

then $f(x)$ is a singleton set for all $x \in M$ and hence f can be seen as a map $f : M \to \hat{M}$.

We show that f is an affine map. To do that, we first make a construction for its differential that is analogous to that for f above. Let $x \in M$ and let $\gamma, \omega \in \angle_{x_0}(M,\nabla)$ be such that $\gamma(1) = x$, $\omega(1) = x$ as above, then since $\omega^{-1}.\gamma \in \square_{x_0}(M,\nabla)$,

$$\mathcal{P}_0^1(\gamma)A_0 = \mathcal{P}_0^1(\omega.\omega^{-1})\mathcal{P}_0^1(\gamma)A_0 = \mathcal{P}_0^1((\omega.\omega^{-1}).\gamma)A_0 = \mathcal{P}_0^1(\omega)\mathcal{P}_0^1(\omega^{-1}.\gamma)A_0 = \mathcal{P}_0^1(\omega)A_0,$$

and so

$$A(x) := \{\mathcal{P}_0^1(\gamma)A_0 \mid \gamma \in \angle_{x_0}(M,\nabla),\ \gamma(1) = x\}$$

is a singleton set for every $x \in M$ and thus we can view A as a map $M \to T^*M \otimes T\hat{M}$.

We claim that $\exists f_*|_x = A(x)$ for all $x \in M$. Indeed, for any $\gamma \in \angle_{x_0}(M,\nabla)$, one has $f(\gamma(t)) = \Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(t)$ and so

$$\frac{\mathrm{d}}{\mathrm{d}t}f(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_{A_0}(\gamma)(t) = (\mathcal{P}_0^t(\gamma)A_0)\dot{\gamma}(t) = A(\gamma(t))\dot{\gamma}(t).$$

Then if $X \in T|_xM$, choose $u \in T|_{x_0}M$ such that $\gamma_u(1) = x$ and notice that $\gamma := \gamma_X.\gamma_u$ is a 1-broken geodesic. Thus the above formula gives by letting $t \to \frac{1}{2}+$,

$$f_*(X) = A(x)X,$$

showing also that the differential $f_*|_x$ exists.

To show that f is an affine map, it is enough to show that for any geodesic $\Gamma : [0,1] \to M$ and any vector field $X(t)$ parallel to it, the vector field $f_*(X(t))$ along $f \circ \Gamma$ is parallel.

So choose such $\Gamma = \gamma_v$ and X. Let $u \in T|_{x_0}M$ be such that $\gamma_u(1) = \Gamma(0)$ and notice that $\gamma_{tv}.\gamma_u \in \angle_{x_0}(M,\nabla)$. Then for all $t \in [0,1]$,

$$f(\Gamma(t)) = f((\gamma_{tv}.\gamma_u)(1)) = \Lambda_{A_0}(\gamma_{tv}.\gamma_u)(1) = \Lambda_{\mathcal{P}_0^1(\gamma_u)A_0}(\Gamma)(t),$$

where the right hand side is a geodesic by Lemma 2.8. Moreover,

$$f_*(X(t)) = A((\gamma_{tv}.\gamma_u)(1))X(t) = (\mathcal{P}_0^1(\gamma_{tv}.\gamma_u)A_0)X(t) = \big(\mathcal{P}_0^t(\Gamma)\mathcal{P}_0^1(\gamma_u)A_0\big)X(t),$$

which, by using $P_t^0(\Gamma)X(t) = X(0)$ and $\Lambda_{\mathcal{P}_0^1(\gamma_u)A_0}(\Gamma) = f \circ \Gamma$, simplifies to

$$f_*(X(t)) = P_0^t(f \circ \Gamma)\big((\mathcal{P}_0^1(\gamma_u)A_0)X(0)\big).$$

Thus $t \mapsto f_*(X(t))$ is the parallel transport of $(\mathcal{P}_0^1(\gamma_u)A_0)X(0)$ along $f \circ \Gamma$ and the proof is finished.

□

We now give the reformulation of the C-A-H Theorem 2.10. The wquivalence of (i),(ii), (v),(vi),(vii),(ix) can essentially be found in [11], the Global C-A-H Theorem 4.47.

Theorem 5.2 Suppose (M,g), $(\hat{M},\hat{g})$ are complete Riemannian manifolds of the same dimension, $\dim M = \dim \hat{M}$, M simply connected and let $A_0 \in T|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ be and infinitesimal isometry. Let $\nabla, \hat{\nabla}$ be the Levi-Civita connections of (M,g), $(\hat{M},\hat{g})$, respectively. Then the following are equivalent (for the sake of clarity we write $\mathcal{P}_a^b(\gamma)$ instead of $(\mathcal{P}^{(\nabla,\hat{\nabla})})_a^b(\gamma)$):

(i) There exists a Riemannian covering map $f : M \to \hat{M}$ such that $f_*|_{x_0} = A_0$;

(ii) For all $\gamma \in \angle_{x_0}(M,\nabla)$ one has $\mathcal{R}^{(\nabla,\hat{\nabla})}_{\mathcal{P}_0^1(\gamma)A_0} = 0$;

(iii) $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\Omega_{x_0}(M)) \subset \Omega_{\hat{x}_0}(\hat{M})$;

(iv) $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\triangle^2_{x_0}(M,\nabla)) \subset \triangle^2_{\hat{x}_0}(\hat{M},\hat{\nabla})$;

(v) $\mathcal{P}_0^1(\gamma)A_0 = A_0$ for all $\gamma \in \Omega_{x_0}(M)$;

(vi) $\mathcal{P}_0^1(\gamma)A_0 = A_0$ for all $\gamma \in \triangle^2_{x_0}(M,\nabla)$;

(vii) $\mathcal{P}_0^1(\gamma)A_0 = A_0$ for all $\gamma \in \square_{x_0}(M,\nabla)$;

(viii) There exist points $x_1 \in M$, $\hat{x}_1 \in \hat{M}$ such that (pw.='piecewise')

$$\forall \gamma : [0,1] \to M \text{ pw. smooth}, \ \gamma(0) = x_0, \ \gamma(1) = x_1 \quad \Longrightarrow \quad \Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(1) = \hat{x}_1.$$

(ix) If $\gamma, \omega : [0,1] \to M$ are piecewise smooth, $\gamma(0) = \omega(0) = x_0$ and $\gamma(1) = \omega(1)$, then $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(1) = \Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\omega)(1)$.

Proof. We write $\gamma_u(t) = \exp_x^{\nabla}(tu)$ if $u \in T|_xM$ and $\hat{\gamma}_{\hat{u}}(t) = \exp_{\hat{x}}^{\hat{\nabla}}(t\hat{u})$ if $\hat{u} \in T|_{\hat{x}}\hat{M}$, $t \in [0,1]$.

We will do the following four cycles of deductions: (i) $\Longleftrightarrow$ (ii) and (i) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) and (i) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ix) $\Rightarrow$ (viii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Since f_* is a local isometry and $f_*|_{x_0} = A_0$, we have $\Lambda_{A_0}(\gamma) = f \circ \gamma$, $f_*|_{\gamma(1)} = \mathcal{P}_0^1(\gamma)A_0$ and hence if $X, Y, Z \in T|_{\gamma(1)}M$,

$$\mathcal{R}_{\mathcal{P}_0^1(\gamma)A_0}(X,Y)Z = f_* R^{\nabla}(X,Y)Z - R^{\hat{\nabla}}(f_*X, f_*Y)Z = 0.$$

(ii) $\Rightarrow$ (i): C-A-H Theorem 2.10

(i) $\Rightarrow$ (v): Again, since f_* is a local isometry and $f_*|_{x_0} = A_0$, then $\mathcal{P}_0^1(\gamma)A_0 = f_*|_{\gamma(1)}$ for any piecewise smooth γ. In particular, $\gamma \in \Omega_{x_0}(M)$ implies $\mathcal{P}_0^1(\gamma)A_0 = f_*|_{\gamma(1)} = f_*|_{x_0} = A_0$.

(v) $\Rightarrow$ (iii): Since $\mathcal{P}_0^1(\gamma)A_0 : T|_{\gamma(1)}M \to T|_{\Lambda_{A_0}(\gamma)(1)}\hat{M}$, and $A_0 : T|_{x_0}M \to T|_{\hat{x}_0}\hat{M}$, it follows that if $\gamma \in \Omega_{x_0}(M)$ and if $\mathcal{P}_0^1(\gamma)A_0 = A_0$, that $\Lambda_{A_0}(\gamma)(1) = \hat{x}_0$ i.e. $\Lambda_{A_0}(\gamma) \in \Omega_{\hat{x}_0}(\hat{M})$.

(iii) $\Rightarrow$ (iv): Obvious (cf. Lemma 2.8).

(iv) $\Rightarrow$ (i): Theorem 3.1 (and the remark that follows it).

(v) $\Rightarrow$ (vi): Obvious.

(vi) $\Rightarrow$ (vii): Let $\Gamma \in \square_{x_0}(M, \nabla)$. After reparameterizing if necessary (see Lemma 2.8 case (v)), we may assume that Γ is $\gamma_4.\gamma_3.\gamma_2.\gamma_1$ with $\gamma_i : [0,1] \to M$, $i = 1,2,3,4$, geodesics. Let $\rho : [0,1] \to M$ be a geodesic from x_0 to $\gamma_2(1) = \gamma_3(0)$. Then $\gamma := \rho^{-1}.(\gamma_2.\gamma_1) \in \triangle_{x_0}(M,\nabla)$ and $\omega := (\gamma_4.\gamma_3).\rho \in \triangle_{x_0}(M,\nabla)$. Hence by assumption and Lemma 2.8 one has

$$\mathcal{P}_0^1(\Gamma)A_0 = \mathcal{P}_0^1(\gamma_4.\gamma_3)\mathcal{P}_0^1(\gamma_2.\gamma_1)A_0 = \mathcal{P}_0^1(\gamma_4.\gamma_3)\mathcal{P}_0^1(\rho.\rho^{-1})\mathcal{P}_0^1(\gamma_2.\gamma_1)A_0 = \mathcal{P}_0^1(\gamma.\omega)A_0 = A_0.$$

(vii) $\Rightarrow$ (i): By Proposition 5.1 there is an affine map $f : M \to \hat{M}$ such that $f_*|_{x_0} = A_0$. Let $x \in M$ and take some geodesic $\gamma : [0,1] \to M$ from x_0 to x. From the affinity of f and $f_*|_{x_0} = A_0$, it follows that $\mathcal{P}_0^1(\gamma)A_0 = f_*|_{\gamma(1)} = f_*|_x$ and since A_0 is an infinitesimal isometry, then so is $\mathcal{P}_0^1(\gamma)A_0$ and hence f is a local isometry. It follows from a standard result in Riemannian geometry that f is a Riemannian covering map.

(i) $\Rightarrow$ (ix): If $f : M \to \hat{M}$ is a Riemannian covering with $f_*|_{x_0} = A_0$ and γ, ω are as stated, then $\Lambda_{A_0}(\gamma) = f \circ \gamma$, $\Lambda_{A_0}(\omega) = f \circ \omega$ and hence $\Lambda_{A_0}(\gamma)(1) = f(\gamma(1)) = f(\omega(1)) = \Lambda_{A_0}(\omega)(1)$.

(ix) $\Rightarrow$ (viii): Take any point $x_1 \in M$, fix any piecewise smooth path $\omega : [0,1] \to M$ from x_0 to x_1 and set $\hat{x}_1 := \Lambda_{A_0}(\omega)(1)$. Then if $\gamma : [0,1] \to M$ is an arbitrary piecewise smooth path from x_0 to x_1, we have $\gamma(1) = x_1 = \omega(1)$ and hence by the assumption, $\Lambda_{A_0}(\gamma)(1) = \Lambda_{A_0}(\omega)(1) = \hat{x}_1$.

(viii) $\Rightarrow$ (i): Corollary 3.5 (and the remark after it). $\square$

Remark 5.3 (a) Although we don't prove it here, the condition (ii) in the previous theorem (and Eq. (1) in C-A-H Theorem 2.10) can in fact be replaced with

(ii)' For all $\gamma \in \angle_{x_0}(M,\nabla)$ and $X \in T|_{\gamma(1)}M$, one has $\mathcal{R}^{(\nabla,\hat{\nabla})}_{\mathcal{P}_0^1(\gamma)A_0}(\dot{\gamma}(1), X)\dot{\gamma}(1) = 0$.

(b) We point out that the condition (v) (resp. (vi)) is significantly stronger than (iii) (resp. (iv)). To see this, consider the set $Q \subset T^*M \otimes T\hat{M}$ of infinitesimal isometries as a bundle over M (resp. over $\hat{M}$), where the bundle map $\pi_M : Q \to M$ (resp. $\pi_{\hat{M}} : Q \to \hat{M}$) maps A to x (resp. to $\hat{x}$), if $A : T|_x M \to T|_{\hat{x}}\hat{M}$. As a manifold Q has dimension $2n + \frac{n(n-1)}{2}$.

If $\gamma : [0,1] \to M$ is a piecewise smooth path that starts from x_0, then $t \mapsto \mathcal{P}_0^t(\gamma)A_0$, $t \in [0,1]$, is a piecewise smooth path in Q that starts from A_0 and $\Lambda_{A_0}(\gamma)(t) = \pi_{\hat{M}}(\mathcal{P}_0^t(\gamma)A_0)$, $\gamma(t) = \pi_M(\mathcal{P}_0^t(\gamma)A_0)$.

The condition (v) says that if γ is a loop of M based at x_0, then $\mathcal{P}_0^t(\gamma)A_0$ is a loop of Q based at A_0 (and therefore automatically $\Lambda_{A_0}(\gamma)$ is a loop of $\hat{M}$ based at $\hat{x}_0$). In other words,

$$\{\mathcal{P}_0^1(\gamma)A_0 \mid \gamma \in \Omega_{x_0}(M)\} = \{A_0\}.$$

On the other hand, condition (iii) demands that for any loop γ of M based at x_0, the path $\mathcal{P}_0^t(\gamma)A_0$ comes back to the set fiber $\pi_{\hat{M}}^{-1}(\hat{x}_0)$ (and of course to $\pi_M^{-1}(x_0)$), where it started from i.e.

$$\{\mathcal{P}_0^1(\gamma)A_0 \mid \gamma \in \Omega_{x_0}(M)\} \subset \pi_M^{-1}(x_0) \cap \pi_{\hat{M}}^{-1}(\hat{x}_0).$$

The set $\pi_M^{-1}(x_0) \cap \pi_{\hat{M}}^{-1}(\hat{x}_0)$ is $\frac{n(n-1)}{2}$ dimensional in contrast to $\{A_0\}$ which is 0-dimensional. This can be seen as an illustration of the stringency of condition (v) with respect to (iii).

(c) It is an open problem to determine if actually there is a weaker version of (vii) i.e. if (i)-(ix) are equivalent to the following: $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\square_{x_0}(M,\nabla)) \subset \square_{\hat{x}_0}(\hat{M},\hat{\nabla})$. See also Remark 4.7.

(d) In (viii) the assumption that γ be piecewise smooth can be replaced by the assumption that it be 6-times broken geodesic (see Corollary 3.5).

(d) The condition (ix) can be replaced by the assumption that γ, ω be 1-broken geodesics.

To see this, we use the argument from [11] which is essentially the same as for Proposition 5.1 (but in Riemannian setting). For any $x \in M$, the set

$$\{\Lambda_{A_0}(\gamma)(1) \mid \gamma \in \angle_{x_0}(M,\nabla),\ \gamma(1) = x\}$$

is a singleton set by assumption, so one may define $f(x)$ to be its unique element. This defines $f : M \to \hat{M}$. If $X \in T|_x M$, let $\omega : [0,1] \to M$ be any geodesic from x_0 to x. Then $\gamma_{tX}.\omega \in \angle_{x_0}(M,\nabla)$, $\gamma_{tX}.\omega(1) = \gamma_X(t)$, and so

$$f(\gamma_X(t)) = \Lambda_{A_0}(\gamma_{tX}.\omega)(1) = \Lambda_{\mathcal{P}_0^1(\omega)A_0}(\gamma_X)(t) = \hat{\gamma}_{(\mathcal{P}_0^1(\omega)A_0)X}(t).$$

This implies that f is differentiable at x and $f_*(X) = (\mathcal{P}_0^1(\omega)A_0)X$. Since $\mathcal{P}_0^1(\omega)A_0$ is an infinitesimal isometry, this implies that f is a local Riemannian isometry (the smoothness of f is easily established) and therefore a Riemannian covering map, i.e. we arrive at case (i).

(e) We also remark that the condition (ix) is much stronger than (viii). To see this, observe that (viii) can be written in the following way that resembles more condition (ix):

(viii) There exist a point $x_1 \in M$ such that if $\gamma, \omega : [0,1] \to M$ are piecewise smooth, $\gamma(0) = \omega(0) = x_0$ and $\gamma(1) = \omega(1) = x_1$, then $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma)(1) = \Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\omega)(1)$.

To put it another way, in (ix) the endpoints $\gamma(1)$ of the curves γ are allowed to move freely on M while in (viii) one only uses curves γ whose endpoints $\gamma(1)$ are fixed to the pre-given point x_1.

Remark 5.4 In [10] the following local version of C-A-H Theorem was proven in the context of affine manifolds: Let (M, ∇), $(\hat{M}, \hat{\nabla})$ be affine manifolds (possibly of different dimensions), let $A_0 \in T^*|_{x_0}M \otimes T|_{\hat{x}_0}\hat{M}$ and suppose $U \subset T|_{x_0}M$ is an open set containing the origin such that $\exp^{\nabla}_{x_0}|_U$ is a diffeomorphism onto its image and that $\exp^{\hat{\nabla}}_{\hat{x}_0}$ is defined on $A_0(U)$. If

$$\mathcal{R}^{(\nabla,\hat{\nabla})}_{\mathcal{P}^1_0(\gamma_u)A_0}(\dot{\gamma}_u(1), X)Y = 0, \quad \mathcal{T}^{(\nabla,\hat{\nabla})}_{\mathcal{P}^1_0(\gamma_u)A_0}(\dot{\gamma}_u(1), X) = 0, \tag{17}$$

for all $u \in U$ and $X, Y \in T|_{\gamma_u(1)}M$, then $\exp^{\hat{\nabla}}_{\hat{x}_0} \circ A_0 \circ (\exp^{\nabla}_{x_0}|_U)^{-1} : U \to \hat{M}$ is an affine map.

We point out that the conclusion (8) of Proposition 4.1 is not enough to invoke this local of C-A-H Theorem in the general setting of affine manifolds, since (8) gives (17) only in the special case where $Y = \dot{\gamma}_u(1)$. It is an open question whether one is able to reach the former condition in (17) from the assumptions of Proposition 4.1.

6 An Application of the Main Result

Recall that the affine group $\mathrm{Aff}(V)$ of a vector space V is $\mathrm{GL}(V) \times V$ as a set and it is equipped with a group multiplication $\star$ given by

$$(A, v) \star (B, w) := (AB, Aw + v), \quad (A, v), (B, w) \in \mathrm{Aff}(V).$$

Also, there is a natural action $\star$ of $\mathrm{Aff}(V)$ on V given by

$$(A, v) \star w := Aw + v, \quad (A, v) \in \mathrm{Aff}(V), \ w \in V.$$

Recall also that if (M, ∇) is an affine manifold and $x \in M$, then its *affine holonomy group* $\mathcal{A}_x$ at x is a subgroup of the affine group $\mathrm{Aff}(T|_x M)$ given by

$$\mathcal{A}_x = \Big\{ \Big(P^0_1(\gamma), \int_0^1 P^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s \Big) \ \Big| \ \gamma \in \Omega_x(M) \Big\}.$$

As an application of Theorem 3.1 we will give a different proof of Theorem IV.7.2 in [8].

Theorem 6.1 Suppose (M, g) is a simply connected, complete Riemannian manifold and $x \in M$. If the affine holonomy group $\mathcal{A}_x$ has a fixed point $W \in T|_x M$, then (M, g) is isometric to the Euclidean space.

Proof. Suppose $W \in T|_x M$ is a fixed point of $\mathcal{A}_x$. Then for all $\gamma \in \Omega_x(M)$ one has $W = \Big(P^0_1(\gamma), \int_0^1 P^0_s(\gamma)\dot{\gamma}(s)\mathrm{d}s\Big) \star W$. Write $\gamma_W : [0, 1] \to M$ for the geodesic with $\dot{\gamma}_W(0) = W$ and define $x_0 := \gamma_W(1)$.

Then if $\omega \in \Omega_{x_0}(M)$, it follows that $\gamma_W^{-1}.(\omega.\gamma_W) \in \Omega_x(M)$,

$$W = \Big(P^0_1(\gamma_W^{-1}.(\omega.\gamma_W)), \int_0^1 P^0_s(\gamma_W^{-1}.(\omega.\gamma_W))\frac{\mathrm{d}}{\mathrm{d}s}(\gamma_W^{-1}.(\omega.\gamma_W))(s)\mathrm{d}s\Big) \star W$$

i.e. if $W' := P^0_1(\gamma_W^{-1})W + \int_0^1 P^0_s(\gamma_W^{-1})\frac{\mathrm{d}}{\mathrm{d}s}\gamma_W^{-1}(s)\mathrm{d}s$,

$$W' = \Big(P^0_1(\omega), \int_0^1 P^0_s(\omega)\dot{\omega}(s)\mathrm{d}s\Big) \star W'.$$

But

$$W' = P_0^1(\gamma_W)W - \int_0^1 P_{1-s}^1(\gamma_W)\dot{\gamma}_W(1-s)\mathrm{d}s = \dot{\gamma}_W(1) - \int_0^1 \dot{\gamma}_W(1)\mathrm{d}s = 0,$$

so one has

$$0 = \int_0^1 P_s^0(\omega)\dot{\omega}(s)\mathrm{d}s, \quad \forall \omega \in \Omega_{x_0}(M). \tag{18}$$

Let A_0 be the identity map $\mathrm{id}_{T|_{x_0}M} : T|_{x_0}M \to T|_{x_0}M$ and define $(\hat{M}, \hat{g}) := (T|_{x_0}M, g|_{T|_{x_0}M})$ and $\hat{x}_0 := 0$, the origin of $T|_{x_0}M$. Then using the natural identification of $T|_{\hat{x}_0}\hat{M} = T|_0(T|_{x_0}M)$ with $T|_{x_0}M$, one sees that A_0 is an infinitesimal isometry. For any piecewise smooth $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$ we obviously have

$$\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\gamma) = \int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\mathrm{d}s,$$

with $\nabla, \hat{\nabla}$ the Levi-Civita connections of (M,g), $(\hat{M},\hat{g})$, respectively. The above equation (18) shows that $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\omega)(1) = 0 = \hat{x}_0$ for all $\omega \in \Omega_{x_0}(M)$ i.e. $\Lambda_{A_0}^{(\nabla,\hat{\nabla})}(\Omega_{x_0}(M)) \subset \Omega_{\hat{x}_0}(\hat{M})$ and thus one may invoke Theorem 3.1 (see also the remark following the theorem) to obtain a Riemannian covering $f : M \to \hat{M}$. Since $(\hat{M},\hat{g})$ is an Euclidean space and in particular simply connected, it follows that f is an isometry from (M,g) to the Euclidean space. This completes the proof. $\square$

Remark 6.2 The above result is used e.g. to determine all the possible affine Riemannian holonomy groups from the usual (linear) holonomy groups (see [8]). Moreover, the affine Riemannian holonomy group turn out to determine the orbits of the the control system associated to the rolling (without slipping and spinning) of a Riemannian manifold onto its tangent plane (see [5]).

Acknowledgements

The work of the author is parto of his PhD thesis and is supported by Finnish Academy of Sciences and Letters, KAUTE foundation and Institut français de Finlande. The author also wants to thank his advisor Y. Chitour for having suggested the problem.

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D Rolling of Manifolds without Spinning

(to appear in Journal of Dynamical and Control Systems, Vol. 19 (2013), No. 1 (January))

Rolling of Manifolds without Spinning

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February 28, 2012

Abstract

The control model of rolling of a Riemannian manifold (M, g) onto another one $(\hat{M}, \hat{g})$ consists of a state space Q of relative orientations (isometric linear maps) between their tangent spaces equipped with a so-called *rolling distribution* $\mathcal{D}_{\mathrm{R}}$, which models the natural constraints of no-spinning and no-slipping of the rolling motion. It turns out that the distribution $\mathcal{D}_{\mathrm{R}}$ can be built as a sub-distribution of a so-called *no-spinning distribution* $\mathcal{D}_{\overline{\nabla}}$ on Q that models only the no-spinning constraint of the rolling motion. One is thus motivated to study the control problem associated to $\mathcal{D}_{\overline{\nabla}}$ and, in particular, the geometry of $\mathcal{D}_{\overline{\nabla}}$-orbits. Moreover, the definition of $\mathcal{D}_{\overline{\nabla}}$ (contrary to the definition of $\mathcal{D}_{\mathrm{R}}$) makes sense in the general context of vector bundles equipped with linear connections.

The purpose of this paper is to study the distribution $\mathcal{D}_{\overline{\nabla}}$ determined by the product connection $\nabla \times \hat{\nabla}$ on a tensor bundle $E^* \otimes \hat{E} \to M \times \hat{M}$ induced by linear connections ∇, $\hat{\nabla}$ on vector bundles $E \to M$, $\hat{E} \to \hat{M}$. We describe completely the orbit structure of $\mathcal{D}_{\overline{\nabla}}$ in terms of the holonomy groups of $\nabla, \hat{\nabla}$ and characterize the integral manifolds of it. Moreover, we describe the general formulas for the Lie brackets of vector fields in $E^* \otimes \hat{E}$ in terms of $\mathcal{D}_{\overline{\nabla}}$ and the vertical tangent distribution of $E^* \otimes \hat{E} \to M \times \hat{M}$.

In the particular case of tangent bundles $TM \to M$, $T\hat{M} \to \hat{M}$ and Levi-Civita connections, we describe in more detail how $\mathcal{D}_{\overline{\nabla}}$ is related to the above mentioned rolling model, where these Lie brackets formulas provide an important tool for the study of controllability of the related control system.

Keywords. Development, Geometry of vector bundles, Riemannian geometry, Linear connections, Rolling of manifolds.

1 Introduction

The rolling model (cf. [4, 15]) consists of two Riemannian manifolds (M, g), $(\hat{M}, \hat{g})$ of the same dimension and the set Q of all linear isometric maps $A : T|_x M \to T|_{\hat{x}} \hat{M}$, so-called relative *orientations*, between their tangents spaces. One associates to this model the problem of *rolling* of M against $\hat{M}$ along some path γ in M and with the given initial relative orientation A_0 between tangent spaces $T|_{\gamma(0)} M$ and $T|_{\hat{x}_0} \hat{M}$ in such a way

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that along the motion, M should not *spin* nor *slip* relative to $\hat{M}$. It turns out that the dynamical condition of *no-spinning* is modelled by a distribution $\mathcal{D}_{\overline{\nabla}}$ on Q that is induced naturally by Levi-Civita connections $\nabla, \hat{\nabla}$ of (M, g), $(\hat{M}, \hat{g})$ and imposing both no-spinning and no-slipping conditions leads to a subdistribution $\mathcal{D}_{\mathrm{R}}$ of $\mathcal{D}_{\overline{\nabla}}$.

Understanding the structure of $\mathcal{D}_{\mathrm{R}}$-orbits has been a subject of active research in recent years, both from the theoretical viewpoint ([2]) and practical applications (cf. [1, 12]). In particular, when the dimension n of M and $\hat{M}$ is equal to two, the structure of the $\mathcal{D}_{\mathrm{R}}$-orbits has been completely characterized (cf. [2]). However, in dimension $n \geq 3$, the problem of becomes considerably more difficult and until last years all available results only considered very specific situations (one of the manifolds is always either an n-sphere or the n-plane) and only focused on characterizing explicitly the $\mathcal{D}_{\mathrm{R}}$ distribution. In [4, 7, 8] the distribution $\mathcal{D}_{\mathrm{R}}$ is studied for two general n-dimensional Riemannian manifolds. In particular, when $n = 3$, the local structure of $\mathcal{D}_{\mathrm{R}}$-orbit is completely characterized in [5]. On the other hand, if one of the manifolds has a non-zero (resp. zero) constant curvature, it is proved in [6] the study of the $\mathcal{D}_{\mathrm{R}}$-orbits is equivalent to the question of determining the holonomy group of a certain vector bundle (resp. affine) connection.

The distribution $\mathcal{D}_{\overline{\nabla}}$ was initially introduced in [4] to build the rolling distribution $\mathcal{D}_{\mathrm{R}}$ and to compute the Lie brackets of vector fields tangent to $\mathcal{D}_{\mathrm{R}}$. This motivates one to study the structure of $\mathcal{D}_{\overline{\nabla}}$-orbits and this is the purpose of the present paper.

It turns out that, since $\mathcal{D}_{\overline{\nabla}}$ is related to the relative parallel transport on manifolds M and $\hat{M}$, one can define it in a more general setting where $\eta : E \to M$ and $\hat{\eta} : \hat{E} \to \hat{M}$ are vector bundles equipped with linear connections $\nabla, \hat{\nabla}$. In this case one obtains a distribution, also called $\mathcal{D}_{\overline{\nabla}}$, in the tensor bundle $E^* \otimes \hat{E}$.

Outline of the paper is the following. Section 3 begins with the description of an induced connection on $E^* \otimes \hat{E}$ and the construction of $\mathcal{D}_{\overline{\nabla}}$ in this general setting is the contents of Sections 3.1. In Sections 3.2 and 3.3 we describe its integral manifolds and orbits, respectively. Then in Section 3.4 we derive the formulas for Lie brackets of vector fields in $E^* \otimes \hat{E}$ with respect to the splitting of $T(E^* \otimes \hat{E})$ determined by $\mathcal{D}_{\overline{\nabla}}$ and the vertical distribution of the bundle $E^* \otimes \hat{E} \to M \times \hat{M}$. These general formulas turn out to be particularly useful when one studies the Lie bracket structure of the sub-distribution $\mathcal{D}_{\mathrm{R}}$ of $\mathcal{D}_{\overline{\nabla}}$ that describes the full rolling motion (no-spin and no-slip) and this is one of the principal reasons for us to derive them.

In Section 4 we restrict to the case where $\eta = \pi_{TM}$ and $\hat{\eta} = \pi_{T\hat{M}}$ are the tangent bundles of M and $\hat{M}$. In this case, on can refine slightly the Lie bracket formula of vector fields tangent to $\mathcal{D}_{\overline{\nabla}}$. We do so in Section 4.1 and derive as an application of it the Lie bracket formula for the vector fields tangent to $\mathcal{D}_{\mathrm{R}}$, whose definition is given there. Section 4.2 studies the controllability of $\mathcal{D}_{\overline{\nabla}}$ in a submanifolds Q of $E^* \otimes \hat{E}$ in the special case where $\nabla, \hat{\nabla}$ are Levi-Civita connections of Riemannian manifolds $M, \hat{M}$ of the same dimension. We also show that while the fibers of Q over $M \times \hat{M}$ are diffeomorphic to the linear Lie-group $\mathrm{SO}(n)$, there is *generically* no principal bundle structure in Q that would render $\mathcal{D}_{\overline{\nabla}}$ to a principal $\mathrm{SO}(n)$-bundle connection if $n \geq 3$. Finally, for the sake of completeness, we discuss in Section 4.3 the rolling model starting from the classical one, and justify (see [4, 7, 15]) that $\mathcal{D}_{\overline{\nabla}}$ (resp. $\mathcal{D}_{\mathrm{R}}$) models the dynamical constraint of no-spinning (resp. no-spinning and no-slipping).

2 Notations

For any sets A, B, C and $U \subset A \times B$ and any map $F : U \to C$, we write U_a and U^b for the sets defined by $\{b \in B \mid (a,b) \in U\}$ and $\{a \in A \mid (a,b) \in U\}$, respectively. Similarly, let $F_a : U_a \to C$ and $F^b : U^b \to C$ be defined by $F_a(b) := F(a,b)$ and $F^b(a) := F(a,b)$ respectively.

If V, W are finite dimensional $\mathbb{R}$-linear spaces, $L : V \to W$ is an $\mathbb{R}$-linear map and $F = (v_i)_{i=1}^{\dim V}$, $G = (w_i)_{i=1}^{\dim W}$ are bases of V, W respectively, the $\dim W \times \dim V$-real matrix corresponding to L w.r.t. the bases F and G is denoted by $\mathcal{M}_{F,G}(L)$.

For any smooth map $\pi : E \to M$ between smooth manifolds E and M, the π-fiber over x is $\pi^{-1}(\{x\}) =: \pi^{-1}(x)$ and it is often written as $E|_x$, when π is clear from the context. The set of smooth sections of π is written as $\Gamma(\pi)$. For a local π-section s, we sometimes write $s|_x$ for its value at $x \in M$ i.e. for $s(x)$. One writes $\mathrm{VF}(M)$ for the set of smooth vector fields on a smooth manifold M i.e., the set of smooth sections of the tangent bundle $\pi_{TM} : TM \to M$.

If $M, \hat{M}$ are smooth manifolds, we will naturally identify $T|_{(x,\hat{x})}(M \times \hat{M})$ with $T|_x M \times T|_{\hat{x}} \hat{M}$, for all $(x, \hat{x}) \in M \times \hat{M}$, and $T(M \times \hat{M})$ with $TM \times T\hat{M}$ without further mention.

If $y \in E$ write $V|_y(\pi)$ for the set of all $Y \in T|_y E$ such that $\pi_*(Y) = 0$. In the case where π is a smooth bundle, the collection of spaces $V|_y(\pi)$, $y \in E$, defines a smooth submanifold $V(\pi)$ of TE and the restriction of $\pi_{TE} : TE \to E$ to $V(\pi)$ is denoted by $\pi_{V(\pi)}$. In this case $\pi_{V(\pi)}$ is a vector subbundle of π_{TE} over E. One says that a smooth distribution $\mathcal{D}$ on E is an *Ehresmann connection* of $\pi : E \to M$, if for all $y \in E$ the map $\pi_*|_y : T|_y E \to T|_{\pi(y)} M$ restricts to a linear isomorphism $\mathcal{D}|_y \to T|_{\pi(y)} M$. In particular then, $T|_y E = \mathcal{D}|_y \oplus V|_y(\pi)$ for all $y \in E$.

If $\rho : N \to M$ is another smooth map, we write $C^\infty(\pi, \rho)$ for the set of smooth maps $F : E \to N$ such that $\rho \circ F = \pi$.

We make the following definitions of vertical derivatives.

Definition 2.1 Let $\rho : H \to N$ be a vector bundle.

(i) If $y \in N$, $u \in H|_y = \rho^{-1}(y)$, write $\nu_\rho|_u$ for the isomorphism

$$\nu_\rho|_u : H|_y \to V|_u(\rho); \quad \nu_\rho|_u(v)(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(u + tv), \quad \forall v \in H|_y,\ \forall f \in C^\infty(H)$$

(ii) Suppose B is a smooth manifold, $\tau : B \to N$ and $F : B \to H$ smooth maps such that $\rho \circ F = \tau$. Then, for $b \in B$ and $\mathcal{V} \in V|_b(\tau)$, we define the vertical derivative of F as

$$\mathcal{V}F := \nu_\rho|_{F(b)}^{-1}(F_* \mathcal{V}) \in H|_{\tau(b)}.$$

We normally omit the index ρ in ν_ρ, when it is clear from the context, and simply write ν instead of ν_ρ. Moreover, it is usually more convenient to write $\nu(v)|_u$ for $\nu|_u(v)$.

For any vector space V, we introduce the notation $\mathcal{T}^k_m(V) := \bigotimes^m V \otimes \bigotimes^k V^*$. As usual, one often suppresses here and in similar notations the parenthesis i.e. writes $\mathcal{T}^k_m V$ for $\mathcal{T}^k_m(V)$. If $\eta : E \to M$ is a vector bundle, then the bundle of (k,m)-tensors of η is written as $\eta_{\mathcal{T}^k_M} : \mathcal{T}^k_m E \to M$ and it's fiber over $x \in M$ is $\mathcal{T}^k_m E|_x$. In the particular case of the tangent bundle $\pi_{TM} : TM \to M$, we write $\pi_{T^k_m M} := (\pi_{TM})_{\mathcal{T}^k_m}$ and $T^k_m M := \mathcal{T}^k_m(TM)$.

We denote by $\Omega_x(M)$, where $x \in M$, the set of all piecewise smooth loops $[0,1] \to M$ of M based at x. If ∇ is a linear connection on $\eta : E \to M$, we write R^∇ for the curvature tensor of ∇ and $(P^\nabla)_0^t(\gamma) : E|_{\gamma(0)} \to E|_{\gamma(t)}$ for the parallel transport along a curve γ in M. The holonomy group of ∇ at $x \in M$ is written as $H^\nabla|_x = \{(P^\nabla)_0^1(\gamma) \mid \gamma \in \Omega_x(M)\}$ and its Lie algebra as $\mathfrak{h}^\nabla|_x$. If ∇ is a connection on $\pi_{TM} : TM \to M$, we write T^∇ for its torsion tensor.

If $\hat{\eta} : \hat{E} \to \hat{M}$ is another vector bundle and one forms the product bundle $\eta \times \hat{\eta} := \eta \times \hat{\eta} : E \times \hat{E} \to M \times \hat{M}$, then the tensor product bundle $(\eta \times \hat{\eta})_{\mathcal{T}^{k+\hat{k}}_{m+\hat{m}}}$ contains a subbundle $\eta_{\mathcal{T}^k_m} \otimes \hat{\eta}_{\mathcal{T}^{\hat{k}}_{\hat{m}}} : \mathcal{T}^k_m E \otimes \mathcal{T}^{\hat{k}}_{\hat{m}} \hat{E} \to M \times \hat{M}$ whose fiber over $(x, \hat{x}) \in M \times \hat{M}$ is $\mathcal{T}^k_m E|_x \otimes \mathcal{T}^{\hat{k}}_{\hat{m}} \hat{E}|_{\hat{x}}$.

If ∇ and $\hat{\nabla}$ are linear connections on these vector bundles $\eta, \hat{\eta}$, respectively, then they induce a product connection $\nabla \times \hat{\nabla}$ on the bundle $\eta \times \hat{\eta}$, which then induces in the usual way a linear connection on $(\eta \times \hat{\eta})_{\mathcal{T}^k_m}$ for any k, m which we still call $\nabla \times \hat{\nabla}$. Respect to this connection the subbundles $\eta_{\mathcal{T}^k_m} \otimes \hat{\eta}_{\mathcal{T}^{\hat{k}}_{\hat{m}}}$, for any $k, \hat{k}, m, \hat{m}$ are parallel and thus $\nabla \times \hat{\nabla}$ restricts to a connection on them, also called $\nabla \times \hat{\nabla}$. In the next section we will recall how to define this on $\eta_{\mathcal{T}^1_0} \otimes \hat{\eta}_{\mathcal{T}^0_1}$.

If $\mathcal{D}$ be a smooth distribution of constant rank on a manifold M one says that a piecewise smooth path $\gamma : [0,1] \to \mathcal{D}$ is tangent to $\mathcal{D}$ if $\dot{\gamma}(t) \in \mathcal{D}|_{\gamma(t)}$ for all t for which the derivative exists. For $x_0 \in M$ one defines the $\mathcal{D}$-orbit through x_0 to be the set

$$\mathcal{O}_{\mathcal{D}}(x_0) = \{\gamma(1) \mid \gamma : [0,1] \to M \text{ piecewise smooth path, tangent to } \mathcal{D},\ \gamma(0) = x_0\}.$$

By the Orbit Theorem (see [3]), it follows that $\mathcal{O}_{\mathcal{D}}(x_0)$ is an immersed smooth submanifold of M containing x_0 with the property that if $f : N \to M$ is a smooth map with $f(N) \subset \mathcal{O}_{\mathcal{D}}(x_0)$, then $f : N \to \mathcal{O}_{\mathcal{D}}(x_0)$ is smooth. One says that $\mathcal{D}$ is (completely) *controllable* in M, if for some (and hence every) point $x_0 \in M$, one has $\mathcal{O}_{\mathcal{D}}(x_0) = M$.

3 Study of an Induced Connection in $E^* \otimes \hat{E}$

In this section, let $\eta : E \to M$ and $\hat{\eta} : \hat{E} \to \hat{M}$ be smooth vector bundles and write $n = \dim M$, $\hat{n} = \dim \hat{M}$, $r = \text{rank } E$, $\hat{r} = \text{rank } \hat{E}$. We will use $\overline{\eta}$ to denote $\eta_{\mathcal{T}^1_0} \otimes \hat{\eta}_{\mathcal{T}^0_1}$ and $E^* \otimes \hat{E}$ for its total space $\mathcal{T}^1_0 E \otimes \mathcal{T}^0_1 \hat{E}$. The fiber over $(x, \hat{x}) \in M \times \hat{M}$ is $E^*|_x \otimes \hat{E}|_{\hat{x}}$. To make it easier to keep track of the base point, we also make the convention the write often $(x, \hat{x}; A)$ for a point $A \in E^*|_x \otimes \hat{E}|_{\hat{x}}$.

Remark 3.1 Since $E^*|_x \otimes \hat{E}|_{\hat{x}}$ is nothing else than $\mathcal{L}(E|_x, \hat{E}|_{\hat{x}})$, the set of linear maps $E|_x \to \hat{E}|_{\hat{x}}$, one also uses the notation $\mathcal{L}(E, \hat{E})$ for $E^* \otimes \hat{E}$ in the literature.

We will assume from now on, if not otherwise mentioned, that η and $\hat{\eta}$ come equipped with linear connections ∇ and $\hat{\nabla}$, resp. These naturally induce a connection $\overline{\nabla} := \nabla \times \hat{\nabla}$ onto $\overline{\eta}$ as explained in the last section. We recall explicitly in the next definition how this is defined.

Definition 3.2 The connection $\overline{\nabla} := \nabla \times \hat{\nabla}$ on $E^* \otimes \hat{E}$ induced by $\nabla, \hat{\nabla}$ is defined by

$$(\overline{\nabla}_{(X,\hat{X})} A)\xi := \hat{\nabla}_{\hat{X}}(A\xi) - A\nabla_X \xi, \tag{1}$$

for $A \in \Gamma(\overline{\eta})$, $\xi \in \Gamma(\eta)$, $X \in \text{VF}(M)$, $\hat{X} \in \text{VF}(\hat{M})$.

To be more precise, if $X \in T|_{x_0} M$, $\hat{X} \in T|_{\hat{x}_0} \hat{M}$ and if γ, $\hat{\gamma}$ are any smooth curves in $M, \hat{M}$, resp., such that $\dot{\gamma}(0) = X$, $\dot{\hat{\gamma}}(0) = \hat{X}$, then $t \mapsto A|_{(\gamma(t), \hat{\gamma}(t))} \xi|_{\gamma(t)}$ is a curve in $\hat{E}$ above $\hat{\gamma}$ and the term $\hat{\nabla}_{\hat{X}}(A\xi)$ on the right hand side means $\hat{\nabla}_{\hat{X}}\big(A|_{(\gamma(\cdot), \hat{\gamma}(\cdot))} \xi|_{\gamma(\cdot)}\big)$.

Remark 3.3 To see that $\overline{\nabla}$ is a well defined linear connection for $\overline{\eta}$, it is enough to prove that for $f \in C^\infty(M)$ such that $f(x_0) = 0$ and any $\lambda \in \Gamma(\eta)$, the right hand side of (1) will vanish for $\xi := f\lambda$. Indeed this is the case, since

$$
\begin{aligned}
&\hat{\nabla}_{\hat{X}}(Af\lambda) - A|_{(x_0,\hat{x}_0)}\nabla_X(f\lambda) \\
=&\hat{\nabla}_{\hat{X}}\big(f(\gamma(\cdot))A|_{(\gamma(\cdot),\hat{\gamma}(\cdot))}\lambda|_{\gamma(\cdot)}\big) - f(x_0)A|_{(x_0,\hat{x}_0)}\nabla_X\lambda - X(f)A|_{(x_0,\hat{x}_0)}\lambda|_{x_0} \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(\gamma(t))A|_{(x_0,\hat{x}_0)}\lambda|_{x_0} + f(x_0)\hat{\nabla}_{\hat{X}}(A\lambda) - 0 - X(f)A|_{(x_0,\hat{x}_0)}\lambda|_{x_0} \\
=&0,
\end{aligned}
$$

where $\gamma, \hat{\gamma}$ were as above.

Remark 3.4 We will also use the symbol $\overline{\nabla}$ to denote the connection $\nabla \times \hat{\nabla}$ on any tensor bundle $(\eta \times \hat{\eta})_{\mathcal{T}^k_m}$ as well as $\eta_{\mathcal{T}^k_m} \otimes \hat{\eta}_{\mathcal{T}^{\hat{k}}_{\hat{m}}}$.

3.1 Introduction of the Distribution $\mathcal{D}_{\overline{\nabla}}$ on $E^* \otimes \hat{E}$

Next definition introduces an Ehresmann connection for the bundle $\overline{\eta} : E^* \otimes \hat{E} \to M \times \hat{M}$ as well as the related lift map.

Definition 3.5 Let $(X, \hat{X}) \in T|_x M \times T|_{\hat{x}}\hat{M}$ and $q = (x, \hat{x}; A) \in E^* \otimes \hat{E}$. One defines the $\mathscr{L}_{\overline{\nabla}}$*-lift* of $(X, \hat{X})$ as the tangent vector $\mathscr{L}_{\overline{\nabla}}(X, \hat{X})|_q \in T|_q(E^* \otimes \hat{E})$ given by

$$
\mathscr{L}_{\overline{\nabla}}(X, \hat{X})|_q := \frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big((P^{\overline{\nabla}})^t_0(\overline{\gamma})A\big),
$$

where $\overline{\gamma}$ is any smooth curve in $M \times \hat{M}$ with $\dot{\overline{\gamma}}(0) = (X, \hat{X})$. Moreover, one defines the subspace $\mathcal{D}_{\overline{\nabla}}|_q$ of $T|_q(E^* \otimes \hat{E})$ by

$$
\mathcal{D}_{\overline{\nabla}}|_q := \mathscr{L}_{\overline{\nabla}}(T|_x M \times T|_{\hat{x}}\hat{M})|_q.
$$

We collect some basic observations in a lemma. The easy proof is omitted here.

Lemma 3.6 (i) If one writes $\overline{\gamma} = (\gamma, \hat{\gamma})$, then

$$
(P^{\overline{\nabla}})^t_0(\overline{\gamma})A = (P^{\hat{\nabla}})^t_0(\hat{\gamma}) \circ A \circ (P^{\nabla})^0_t(\gamma),
$$

(ii) For every $q = (x, \hat{x}; A) \in E^* \otimes \hat{E}$, the plane $\mathcal{D}_{\overline{\nabla}}|_q$ has dimension $n + \hat{n}$ and $\overline{\eta}_*|_{\mathcal{D}_{\overline{\nabla}}|_q} : \mathcal{D}_{\overline{\nabla}}|_q \to T|_x M \times T|_{\hat{x}}\hat{M}$ is an isomorphism.

(iii) If $\overline{X} \in \mathrm{VF}(M \times \hat{M})$, then the map

$$
E^* \otimes \hat{E} \to T(E^* \otimes \hat{E}); \quad q = (x, \hat{x}; A) \mapsto \mathscr{L}_{\overline{\nabla}}(\overline{X}|_{(x,\hat{x})})|_q
$$

is smooth. In particular, the distribution $q \mapsto \mathcal{D}_{\overline{\nabla}}|_q$ in $E^* \otimes \hat{E}$ is smooth.

The following basic formula for the lift $\mathscr{L}_{\overline{\nabla}}$ will be useful.

Proposition 3.7 For $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ and $A \in \Gamma(\overline{\eta})$, we have

$$\mathscr{L}_{\overline{\nabla}}(\overline{X})|_{A|_{(x,\hat{x})}} = A_*(\overline{X}) - \nu\big(\overline{\nabla}_{\overline{X}}A\big)|_{A|_{(x,\hat{x})}}, \tag{2}$$

where A_* is the map $T(M \times \hat{M}) \to T(E^* \otimes \hat{E})$.

Proof. First, we prove that if $\rho : H \to N$ is a vector bundle and $\tau, \lambda : [0,1] \to H$ are smooth curves with $\rho \circ \tau = \rho \circ \lambda$, then the tangent vector to the curve $t \mapsto \lambda(t) + t\tau(t)$ in H at $t = 0$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|_0(\lambda(t) + t\tau(t)) = \dot{\lambda}(0) + \nu_\rho(\tau(0))|_{\lambda(0)}.$$

Indeed, if $f \in C^\infty(H)$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(\lambda(t) + t\tau(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(\lambda(t)) = \lim_{t\to 0}\frac{f(\lambda(t) + t\tau(t)) - f(\lambda(t))}{t}\\ = \lim_{t\to 0}\frac{1}{t}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} f(\lambda(t) + s\tau(t))\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}s} f(\lambda(0) + s\tau(0)) = \nu_\rho(\tau(0))|_{\lambda(0)} f.\end{aligned}$$

Now that this has been established, we choose smooth path $\overline{\gamma} : [-1,1] \to M \times \hat{M}$ such that $\dot{\overline{\gamma}}(0) = \overline{X}$. It being clear that (see e.g. [14], p.29)

$$(P^{\overline{\nabla}})^0_t(\overline{\gamma})(A|_{\overline{\gamma}(t)}) = A|_{(x,\hat{x})} + t\overline{\nabla}_{\overline{X}}A + t^2F(t),$$

for a smooth $F : [-1,1] \to \mathbb{R}$, we get by using the definition of $\mathscr{L}_{\overline{\nabla}}$ and the formula that we just proved above (take there $\lambda(t) = A|_{\overline{\gamma}(t)}$, $\tau(t) = -(P^{\overline{\nabla}})^t_0(\overline{\nabla}_{\overline{X}}A + tF(t))$),

$$\begin{aligned}\mathscr{L}_{\overline{\nabla}}(\overline{X})|_{A|_{(x,\hat{x})}} =& \frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big((P^{\overline{\nabla}})^t_0 A|_{(x,\hat{x})}\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0\Big(A|_{\overline{\gamma}(t)} - t(P^{\overline{\nabla}})^t_0\big(\overline{\nabla}_{\overline{X}}A + tF(t)\big)\Big)\\ =& A_*(\overline{X}) - \nu(\overline{\nabla}_{\overline{X}}A)|_{A|_{(x,\hat{x})}}.\end{aligned}$$

□

Remark 3.8 Of course, Definition 3.5 makes sense in any bundle $\eta_{T^k_m} \otimes \hat{\eta}_{T^{\hat{k}}_{\hat{m}}}$ and $(\eta \times \hat{\eta})_{T^k_m}$. Also, it is clear that Proposition 3.7 holds true in these settings.

3.2 Integrability of $\mathcal{D}_{\overline{\nabla}}$

Proposition 3.9 Let $X, Y \in \mathrm{VF}(M)$, $\hat{X}, \hat{Y} \in \mathrm{VF}(\hat{M})$. Then at $q = (x, \hat{x}; A) \in E^* \otimes \hat{E}$ one has

$$[\mathscr{L}_{\overline{\nabla}}(X,\hat{X}), \mathscr{L}_{\overline{\nabla}}(Y,\hat{Y})]|_q = \mathscr{L}_{\overline{\nabla}}([X,Y],[\hat{X},\hat{Y}])|_q + \nu\big(AR^{\nabla}(X,Y) - R^{\hat{\nabla}}(\hat{X},\hat{Y})A\big)|_q.$$

We postpone the proof to next section, just after the proof of Proposition 3.22.

Corollary 3.10 Let $q = (x, \hat{x}; A) \in E^* \otimes \hat{E}$. Then $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is an integral manifold of $\mathcal{D}_{\overline{\nabla}}$ if and only if

$$\mathrm{im}\mathfrak{h}^{\nabla}|_x \subset \ker A \quad \text{and} \quad \mathrm{im}A \subset \ker \mathfrak{h}^{\hat{\nabla}}|_{\hat{x}} \tag{3}$$

Remark 3.11 Formula $\mathrm{im}\mathfrak{h}^{\nabla}|_x \subset \ker A$ means that for any $U \in \mathfrak{h}^{\nabla}$, one has $A \circ U = 0$ On the other hand, $\mathrm{im}A \subset \ker \mathfrak{h}^{\hat{\nabla}}|_{\hat{x}}$ means that for all $\hat{U} \in \mathfrak{h}^{\hat{\nabla}}|_{\hat{x}}$, $\hat{U} \circ A = 0$.

Proof. In the proof, we will use "$\circ$" to denote the composition of linear maps in order to avoid confusion. By Ambrose-Singer theorem (see [10]) one has

$$\mathfrak{h}^{\nabla}|_x = \mathrm{span}\{(P^{\nabla})^0_1(\gamma) \circ R^{\nabla}|_{\gamma(1)}(X,Y) \circ (P^{\nabla})^1_0(\gamma) \mid \gamma : [0,1] \to M; \gamma(0) = x, \\ X, Y \in T|_{\gamma(1)}M\}$$

and similarly for $\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}_0}$. The orbit, on the other hand, is

$$\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q) = \{(P^{\hat{\nabla}})^1_0(\hat{\gamma}) \circ A \circ (P^{\nabla})^0_1(\gamma) \mid (\gamma, \hat{\gamma}) : [0,1] \to M \times \hat{M},\ \gamma(0) = x,\ \hat{\gamma}(0) = \hat{x}\}.$$

If $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is an integral manifold of $\mathcal{D}_{\overline{\nabla}}$, then by Proposition 3.9 one must have

$$A_1 \circ R^{\nabla}(X,Y) - R^{\hat{\nabla}}(\hat{X},\hat{Y}) \circ A_1 = 0,$$

for all $(x_1, \hat{x}_1; A_1) \in \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$, $X, Y \in T|_{x_1}M$, $\hat{X}, \hat{Y} \in T|_{\hat{x}_1}\hat{M}$. Taking first here $X = Y = 0$ and then $\hat{X} = \hat{Y} = 0$ and using the above expression for the orbit, one sees that this is equivalent to

$$A \circ (P^{\nabla})^0_1(\gamma) \circ R^{\nabla}(X,Y) = 0, \qquad \forall \gamma : [0,1] \to M,\ \gamma(0) = x,\ \forall X, Y \in T|_{\gamma(1)}M,$$
$$R^{\hat{\nabla}}(\hat{X},\hat{Y}) \circ (P^{\hat{\nabla}})^1_0(\hat{\gamma}) \circ A = 0, \qquad \forall \hat{\gamma} : [0,1] \to \hat{M},\ \hat{\gamma}(0) = \hat{x},\ \forall \hat{X}, \hat{Y} \in T|_{\hat{\gamma}(1)}\hat{M}.$$

Composing the first (resp. the second) equation from the right (resp. left) with $(P^{\nabla})^1_0(\gamma)$ (resp. $(P^{\hat{\nabla}})^0_1(\hat{\gamma})$) and using the above mentioned Ambrose-Singer theorem, we get (3).

Conversely, assume that (3) holds. Then reversing the argument we just made above, we see that, according to Proposition 3.9 again,

$$[\mathscr{L}_{\overline{\nabla}}(X,\hat{X}), \mathscr{L}_{\overline{\nabla}}(Y,\hat{Y})]|_{q_1} = \mathscr{L}_{\overline{\nabla}}([X,Y],[\hat{X},\hat{Y}])|_{q_1},$$

for all $q_1 \in \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$, $X, Y \in \mathrm{VF}(M)$, $\hat{X}, \hat{Y} \in \mathrm{VF}(\hat{M})$. But by the definition of an orbit, the distribution $\mathcal{D}_{\overline{\nabla}}$ is tangent to $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ and so the above shows that $\mathcal{D}_{\overline{\nabla}}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)}$ is involutive (as a distribution in $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$). This immediately implies that $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is the unique leaf of $\mathcal{D}_{\overline{\nabla}}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)}$ which means that it is an integral manifold of $\mathcal{D}_{\overline{\nabla}}$. End of the proof. □

Corollary 3.12 Suppose $r = \hat{r}$ and $q = (x, \hat{x}; A) \in \mathrm{GL}(E, \hat{E})$. Then $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is an integral manifold of $\mathcal{D}_{\overline{\nabla}}$ if and only if ∇ and $\hat{\nabla}$ are flat, i.e. $R^{\nabla} = 0$ and $R^{\hat{\nabla}} = 0$.

Proof. Indeed $\ker A = \{0\}$, $\mathrm{im}A = T|_xM$, so $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is an integral manifold of $\mathcal{D}_{\overline{\nabla}}$ if and only if $\mathfrak{h}^{\nabla}|_x = 0$ and $\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}} = 0$. By Ambrose-Singer theorem this happens if and only if R^{∇} and $R^{\hat{\nabla}}$ vanish everywhere. □

3.3 The Orbit Structure of $\mathcal{D}_{\overline{\nabla}}$

Proposition 3.13 Let $q = (x, \hat{x}; A) \in E^* \otimes \hat{E}$. Then the fiber over $(x, \hat{x})$ of the orbit $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is given by

$$\big(E^*|_x \otimes \hat{E}|_{\hat{x}}\big) \cap \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q) = H^{\hat{\nabla}}|_{\hat{x}} \circ A \circ H^{\nabla}|_x$$

and its vertical tangent space by

$$V|_q(\overline{\eta}) \cap T|_q\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q) = \nu\big(\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}} \circ A + A \circ \mathfrak{h}^{\nabla}|_x\big)\big|_q.$$

Proof. By Lemma 3.6 (i) and the definition of the orbit $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$, we have

$$\begin{aligned}\big(E^*|_x \otimes \hat{E}|_{\hat{x}}\big) \cap \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q) =&(\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)})^{-1}(x,\hat{x}) \\ =&\{(P^{\hat{\nabla}})_0^1(\hat{\gamma}) \circ A \circ (P^{\nabla})_1^0(\gamma) \mid \gamma \in \Omega_x(M),\ \hat{\gamma} \in \Omega_{\hat{x}}(\hat{M})\}\end{aligned}$$

By the definition of a holonomy group, the right hand side equals $H^{\hat{\nabla}}|_{\hat{x}} \circ A \circ H^{\nabla}|_x$.

To prove the expression for the tangent space $V|_q(\overline{\eta}) \cap T|_q\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ of the fiber $(\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)})^{-1}(x,\hat{x})$, at q, we define a group $G := H^{\hat{\nabla}}|_{\hat{x}} \times H^{\nabla}|_x$ and

$$\mu : G \times (\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)})^{-1}(x,\hat{x}) \to (\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)})^{-1}(x,\hat{x}); \quad \mu((\hat{h},h),A') = \hat{h} \circ A' \circ h^{-1}.$$

By what we have shown, μ is a transitive left G-action on the fiber $(\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)})^{-1}(x,\hat{x})$. Then map $\psi : G \to (\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)})^{-1}(x,\hat{x})$ such that $(\hat{h},h) \mapsto \mu((\hat{h},h),A)$ is a left G-equivariant surjection onto $(\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)})^{-1}(x,\hat{x})$ and thus a submersion, since it necessarily has constant rank. Therefore tangent space $V|_q(\overline{\eta}) \cap T|_q\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ to the fiber is spanned by vectors of the form $\psi_*(\hat{B},B)$ where $\hat{B} \in \mathfrak{h}^{\hat{\nabla}}|_{\hat{x}}$, $B \in \mathfrak{h}^{\nabla}|_x$. Since clearly

$$\psi_*(\hat{B},B) = \nu(\hat{B} \circ A - A \circ B)|_q,$$

claimed description of $V|_q(\overline{\eta}) \cap T|_q\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is established. □

Remark 3.14 This provides another proof of Corollary 3.12. Indeed, $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is an integral manifold of $\mathcal{D}_{\overline{\nabla}}$ if and only if $V|_q(\overline{\eta}) \cap T|_q\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q) = \{0\}$ which, by the previous proposition, is equivalent to $\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}} \circ A + A \circ \mathfrak{h}^{\nabla}|_x = \{0\}$ i.e. $A \circ \mathfrak{h}^{\nabla}|_x = 0$ and $\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}} \circ A = 0$.

Next Proposition allows us to study some obstructions, in the case $r = \hat{r}$, for $\mathcal{D}_{\overline{\nabla}}$ to be a principal bundle connection when restricted to some subbundle of $\overline{\eta}$.

Proposition 3.15 Suppose $r = \hat{r}$ and let $F = (\xi_i)$, $\hat{F} = (\hat{\xi}_i)$ be frames of $E|_{x_0}$ and $\hat{E}|_{\hat{x}_0}$, respectively, and let $H^{\nabla}|_F \subset \mathrm{GL}(r)$, $H^{\hat{\nabla}}|_{\hat{F}} \subset \mathrm{GL}(r)$ be the holonomy groups w.r.t. to these frames. Define $q_0 = (x_0,\hat{x}_0;A_0) \in \mathrm{GL}(E,\hat{E})$ such that $A_0\xi_i = \hat{\xi}_i$, $i = 1,\dots,r$.

Then if $G : \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \to E^* \otimes \hat{E}$ is smooth, $\overline{\eta} \circ G = \overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)}$ and if $G_*\mathcal{D}_{\overline{\nabla}}|_q = \mathcal{D}_{\overline{\nabla}}|_{G(q)}$ for all $q \in \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$, then every element of $H^{\nabla}|_F \cap H^{\hat{\nabla}}|_{\hat{F}}$ commutes with $\mathcal{M}_{F,\hat{F}}(G(A_0)) \in \mathfrak{gl}(r)$.

Proof. Let $q(t) = (\gamma(t),\hat{\gamma}(t);A(t))$ be a piecewise smooth path in $E^* \otimes \hat{E}$ such that $q(0) = q_0$ and $\dot{q}(t) \in \mathcal{D}_{\overline{\nabla}}$ for a.e. t. Then $q(t) \in \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ for all t and the assumptions imply that $\frac{\mathrm{d}}{\mathrm{d}t}(G \circ q)(t) = G_*\dot{q}(t) \in \mathcal{D}_{\overline{\nabla}}$ for a.e. t. Since $\overline{\eta}((G \circ q)(t)) = \overline{\eta}(q(t)) = (\gamma(t),\hat{\gamma}(t))$, it follows that

$$(G \circ q)(t) = \big(\gamma(t),\hat{\gamma}(t);(P^{\overline{\nabla}})_0^t(\gamma,\hat{\gamma})G(A_0)\big)$$

i.e.

$$G\big((P^{\hat{\nabla}})_0^1(\hat{\gamma}) \circ A_0 \circ (P^{\nabla})_1^0(\gamma)\big) = (P^{\hat{\nabla}})_0^1(\hat{\gamma}) \circ G(A_0) \circ (P^{\nabla})_1^0(\gamma).$$

Next take any $B \in H^{\nabla}|_F \cap H^{\hat{\nabla}}|_{\hat{F}}$ and choose loops $\gamma,\hat{\gamma}$ such that $B = \mathcal{M}_{F,F}\big((P^{\nabla})_0^1(\gamma)\big) = \mathcal{M}_{\hat{F},\hat{F}}\big((P^{\hat{\nabla}})_0^1(\hat{\gamma})\big)$. Then by the choice of A_0 w.r.t. $F,\hat{F}$ we have

$$\begin{aligned}\mathcal{M}_{F,\hat{F}}(A_0) = \mathrm{id}_{\mathbb{R}^r} = B\mathrm{id}_{\mathbb{R}^r}B^{-1} =& \mathcal{M}_{\hat{F},\hat{F}}\big((P^{\hat{\nabla}})_0^1(\hat{\gamma})\big)\mathcal{M}_{F,\hat{F}}(A_0)\underbrace{\mathcal{M}_{F,F}\big((P^{\nabla})_1^0(\gamma)\big)}_{=\mathcal{M}_{F,F}\left((P^{\nabla})_0^1(\gamma)\right)^{-1}} \\ =& \mathcal{M}_{F,\hat{F}}\big((P^{\hat{\nabla}})_0^1(\hat{\gamma}) \circ A_0 \circ (P^{\nabla})_1^0(\gamma)\big),\end{aligned}$$

i.e. $A_0 = (P^{\hat{\nabla}})_0^1(\hat{\gamma}) \circ A_0 \circ (P^{\nabla})_1^0(\gamma)$. Evaluating G here and using what was proved above we obtain $G(A_0) = (P^{\hat{\nabla}})_0^1(\hat{\gamma}) \circ G(A_0) \circ (P^{\nabla})_1^0(\gamma)$ i.e.

$$\mathcal{M}_{F,\hat{F}}(G(A_0)) = B\mathcal{M}_{F,\hat{F}}(G(A_0))B^{-1}$$

which is what we wished to prove.

□

3.4 Lie Brackets

It is clear that $T(E^* \otimes \hat{E})$ splits into a direct sum

$$T(E^* \otimes \hat{E}) = \mathcal{D}_{\overline{\nabla}} \oplus V(\overline{\eta}),$$

i.e. $\overline{\eta}_* = \overline{\eta}_*|_{\mathcal{D}_{\overline{\nabla}}} \oplus \pi_{V(\overline{\eta})}$ as vector bundles over $E \otimes \hat{E}$. Therefore any vector field $\mathcal{X} \in \mathrm{VF}(E^* \otimes \hat{E})$ can be written uniquely in the form

$$\mathcal{X}|_q = \mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q + \nu(U(q))|_q,$$

where $\overline{T} \in C^\infty(\overline{\eta}, \pi_{T(M\times\hat{M})})$ and $U \in C^\infty(\overline{\eta}, \overline{\eta})$. This implies that to compute a Lie bracket of two vector fields in $E^* \otimes \hat{E}$, it suffices to know the formulas for Lie brackets of vector fields of the forms $\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot)) : q \mapsto \mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q$ and $\nu(U(\cdot)) : q \mapsto \nu(U(q))|_q$.

To be able to state the bracket formulas in a more invariant form, we make the following definition (recall also Definition 3.5).

Definition 3.16 Let $\mathcal{O} \subset E^* \otimes \hat{E}$ be an immersed submanifold, $q = (x, \hat{x}; A) \in \mathcal{O}$, $\overline{X} \in T|_{(x,\hat{x})}(M \times \hat{M})$ and $S : \mathcal{O} \to \mathcal{T}^k_m(E \times \hat{E})$ be a smooth map. If moreover $\mathscr{L}_{\overline{\nabla}}(\overline{X})|_q \in T|_q\mathcal{O}$, one defines $\mathscr{L}_{\overline{\nabla}}(\overline{X})|_q S \in \mathcal{T}^k_m(E \times \hat{E})|_{(x,\hat{x})}$ by

$$(\mathscr{L}_{\overline{\nabla}}(\overline{X})|_q S)\overline{\omega} := \mathscr{L}_{\overline{\nabla}}(\overline{X})|_q(S\overline{\omega}) - S(q)\overline{\nabla}_{\overline{X}}\overline{\omega}, \qquad (4)$$

for all $\overline{\omega} \in \Gamma(\overline{\eta}_{\mathcal{T}^m_k})$.

Notice that here $\overline{\omega}|_{(x,\hat{x})}$ and $\overline{\nabla}_{\overline{X}}\overline{\omega}$ are elements of $\mathcal{T}^m_k(E \times \hat{E}) = (\mathcal{T}^k_m(E \times \hat{E}))^*$ so $S(q)\overline{\nabla}_{\overline{X}}\overline{\omega} \in \mathbb{R}$ and one may view $\overline{T}\overline{\omega}$ as a smooth map $\mathcal{O} \to \mathbb{R}$; $q' = (x', \hat{x}'; A) \mapsto S(q')\overline{\omega}|_{(x',\hat{x}')}$ to which $\mathscr{L}_{\overline{\nabla}}(\overline{X})|_q$ on the right hand side acts as a tangent vector of $\mathcal{O}$.

Remark 3.17 If $S : \mathcal{O} \to \mathcal{T}^m_k E \otimes \mathcal{T}^{\hat{m}}_{\hat{k}} \hat{E}$, then clearly $\mathscr{L}_{\overline{\nabla}}(\overline{X})|_q S \in \mathcal{T}^m_k E|_x \otimes \mathcal{T}^{\hat{m}}_{\hat{k}} \hat{E}|_{\hat{x}}$.

Before attacking the problem of computation of the brackets, we prove three elementary lemmas.

Lemma 3.18 Let $q_0 = (x_0, \hat{x}_0; A_0) \in E^* \otimes \hat{E}$. Then there exists a local $\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)}$-section $\tilde{A}$ defined around $(x_0, \hat{x}_0)$ such that $\tilde{A}|_{(x_0,\hat{x}_0)} = A$ and $\overline{\nabla}\tilde{A}|_{(x_0,\hat{x}_0)} = 0$.

Proof. Let (U, ϕ), $(\hat{U}, \hat{\phi})$ be charts around x_0 and $\hat{x}_0$ such that $\phi(x_0) = 0$, $\hat{\phi}(\hat{x}_0) = 0$ and $\phi(U) = \mathbb{R}^n$, $\hat{\phi}(\hat{U}) = \mathbb{R}^{\hat{n}}$. For $(x, \hat{x}) \in U \times \hat{U}$ define

$$\tilde{A}|_{(x,\hat{x})} := (P^{\overline{\nabla}})^1_0\Big(t \mapsto (\phi \times \hat{\phi})^{-1}\big(t(\phi \times \hat{\phi})(x, \hat{x})\big)\Big)A_0.$$

Obviously, $\tilde{A}$ is a smooth local section $U \times \hat{U} \to E^* \otimes \hat{E}$ of $\overline{\eta}$ and $\tilde{A}|_{(x_0,\hat{x}_0)} = A_0$. To show that $\overline{\nabla}\tilde{A}|_{(x_0,\hat{x}_0)} = 0$, take any $(X,\hat{X}) \in T|_{(x_0,\hat{x}_0)}(M \times \hat{M})$ and choose a point $(x,\hat{x}) \in U \times \hat{U}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 (\phi \times \hat{\phi})^{-1}\big(t(\phi \times \hat{\phi})(x,\hat{x})\big) = (X,\hat{X}).$$

Such a point $(x,\hat{x})$ exists and is unique since if one makes the usual identification $T|_{(0,0)}(\mathbb{R}^n \times \mathbb{R}^{\hat{n}}) = \mathbb{R}^n \times \mathbb{R}^{\hat{n}}$, then $(\phi \times \hat{\phi})(x,\hat{x}) = (\phi \times \hat{\phi})_*(X,\hat{X})$.

Writing

$$\overline{\gamma}(t) = \overline{\gamma}_{(x,\hat{x})}(t) := (\phi \times \hat{\phi})^{-1}(t(\phi \times \hat{\phi})(x,\hat{x})),$$

we have $\dot{\overline{\gamma}}(0) = (X,\hat{X})$ and

$$\overline{\nabla}_{(X,\hat{X})}\tilde{A} = \frac{\mathrm{d}}{\mathrm{d}s}\big|_0 (P^{\overline{\nabla}})^0_s(\overline{\gamma})\tilde{A}|_{\overline{\gamma}(s)} = \frac{\mathrm{d}}{\mathrm{d}s}\big|_0 (P^{\overline{\nabla}})^0_s(\overline{\gamma})(P^{\overline{\nabla}})^1_0(t \mapsto \overline{\gamma}(ts))A_0 = 0.$$

To see that $\tilde{A}$ is a smooth local section $\overline{\eta}|_{\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)}$ defined on $U \times \hat{U}$ follows by noticing that for $(x,\hat{x}) \in U \times \hat{U}$ and $s \in [0,1]$,

$$\begin{aligned}\tilde{A}|_{\overline{\gamma}_{(x,\hat{x})}(s)} =&(P^{\overline{\nabla}})^1_0\big(t \mapsto \overline{\gamma}_{\overline{\gamma}_{(x,\hat{x})}(s)}(t)\big)A_0 = (P^{\overline{\nabla}})^1_0\big(t \mapsto \overline{\gamma}_{(x,\hat{x})}(ts)\big)A_0 \\ =&(P^{\overline{\nabla}})^s_0\big(t \mapsto \overline{\gamma}_{(x,\hat{x})}(t)\big)A_0\end{aligned}$$

which shows, by definition of $\mathscr{L}_{\overline{\nabla}}$ and $\mathcal{D}_{\overline{\nabla}}$ that for all $s \in [0,1]$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\tilde{A}|_{\overline{\gamma}_{(x,\hat{x})}(s)} = \mathscr{L}_{\overline{\nabla}}\big(\dot{\overline{\gamma}}_{(x,\hat{x})}(s)\big)\big|_{\tilde{A}|_{\overline{\gamma}_{(x,\hat{x})}(s)}} \in \mathcal{D}_{\overline{\nabla}}|_{\tilde{A}|_{\overline{\gamma}_{(x,\hat{x})}(s)}}$$

and so

$$\tilde{A}|_{(x,\hat{x})} = \tilde{A}|_{\overline{\gamma}_{(x,\hat{x})}(1)} \in \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}\big(\tilde{A}|_{\overline{\gamma}_{(x,\hat{x})}(0)}\big) = \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0).$$

Finally, since $\tilde{A}$ is a smooth map $U \times \hat{U} \to E^* \otimes \hat{E}$ whose image is inside $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$, then $\tilde{A}$ is smooth as a map $U \times \hat{U} \to \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$ since the orbit is a weakly embedded submanifold of $E^* \otimes \hat{E}$. □

Lemma 3.19 Let $q = (x,\hat{x};A) \in E^* \otimes \hat{E}$ and $\overline{X} = (X,\hat{X}), \overline{Y} = (Y,\hat{Y}) \in T|_{(x,\hat{x})}(M \times \hat{M})$. Then

$$R^{\overline{\nabla}}(\overline{X},\overline{Y})A = -AR^{\nabla}(X,Y) + R^{\hat{\nabla}}(\hat{X},\hat{Y})A.$$

Proof. Assume that A is of the form $f^* \otimes \hat{f} \in E^* \otimes \hat{E}|_{(x,\hat{x})}$ Then since a curvature is a derivation in tensor algebra, we have

$$\begin{aligned}R^{\overline{\nabla}}(\overline{X},\overline{Y})A =&(R^{\overline{\nabla}}(\overline{X},\overline{Y})f^*) \otimes \hat{f} + f^* \otimes (R^{\overline{\nabla}}(\overline{X},\overline{Y})\hat{f}) \\ =&(R^{\nabla}(X,Y)f^*) \otimes \hat{f} + f^* \otimes (R^{\hat{\nabla}}(\hat{X},\hat{Y})\hat{f}) \\ =&(-f^*R^{\nabla}(X,Y)) \otimes \hat{f} + f^* \otimes (R^{\hat{\nabla}}(\hat{X},\hat{Y})\hat{f}) \\ =&-(f^* \otimes \hat{f})R^{\nabla}(X,Y) + R^{\hat{\nabla}}(\hat{X},\hat{Y})(f^* \otimes \hat{f}) \\ =&-AR^{\nabla}(X,Y) + R^{\hat{\nabla}}(\hat{X},\hat{Y})A,\end{aligned}$$

where $R^\nabla(X,Y)f^* = -f^* R^\nabla(X,Y)$ follows by noticing that since $R^\nabla(X,Y)$ also commutes with contraction $C : E^* \otimes E \to \mathbb{R}$ and acts trivially on $\mathbb{R}$,

$$\begin{aligned} 0 =& R^\nabla(X,Y)(f^*Z) = R^\nabla(X,Y)C(f^*\otimes Z) = C\big(R^\nabla(X,Y)(f^*\otimes Z)\big) \\ =& C\big((R^\nabla(X,Y)f^*)\otimes Z\big) + C\big(f^*\otimes (R^\nabla(X,Y)Z)\big) \\ =& (R^\nabla(X,Y)f^*)Z + f^*(R^\nabla(X,Y)Z) \end{aligned}$$

where $Z \in E|_x$.

The statement for a general $A \in E^* \otimes \hat{E}|_{(x,\hat{x})}$ then follows by linearity. □

We will need one more lemma, which allows us to restrict the formulas obtained for commutators from open subsets of $E^* \otimes \hat{E}$ to submanifolds of $E^* \otimes \hat{E}$.

Lemma 3.20 Let N be a smooth manifold, $\eta : E \to N$ a vector bundle, $\tau : B \to N$ a smooth map, $\mathcal{O} \subset B$ an immersed submanifold and $F : \mathcal{O} \to E$ a smooth map such that $\eta \circ F = \tau|_{\mathcal{O}}$. With a diagram the situation is

$$\begin{tikzcd} B \supset \mathcal{O} \arrow[rr, "F"] \arrow[dr, "\tau"'] & & E \arrow[dl, "\eta"] \\ & N & \end{tikzcd}$$

(i) Then for every $b_0 \in \mathcal{O}$, there exists an open neighbourhood V of b_0 in $\mathcal{O}$, an open neighbourhood $\tilde{V}$ of b_0 in B such that $V \subset \tilde{V}$ and a smooth map $\tilde{F} : \tilde{V} \to E$ such that $\eta \circ \tilde{F} = \tau|_{\tilde{V}}$ and $\tilde{F}|_V = F|_V$. We call $\tilde{F}$ a local extension of F around b_0.

(ii) Suppose $\tau : B \to N$ is also a vector bundle and $\tilde{F}$ is any local extension of F around b_0 as in case (i). Then if $v \in B|_{\tau(b_0)}$ is such that $\nu|_{b_0}(v) \in T|_{b_0}\mathcal{O}$, one has

$$\nu|_{b_0}(v)(F) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \tilde{F}(b_0 + tv) \in E|_{\tau(b_0)},$$

where on the right hand side one views $t \mapsto \tilde{F}(b_0+tv)$ as a map into a fixed (i.e. independent of t) vector space $E|_{F(b_0)}$ and the derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ is just the classical derivative of a vector valued map (and not a tangent vector).

Proof. (i) For a given $b_0 \in \mathcal{O}$, take a neighbourhood W of $y_0 := \tau(b_0)$ in N such that there exists a local frame $v_1,\dots,v_k$ of η defined on W (here $k = \dim E - \dim N$). Since $\eta \circ F = \tau|_{\mathcal{O}}$, it follows that

$$F(b) = \sum_{i=1}^k f_i(b) v_i|_{\tau(b)}, \quad \forall b \in \tau^{-1}(W) \cap \mathcal{O},$$

for some smooth functions $f_i : \tau^{-1}(W)\cap\mathcal{O} \to \mathbb{R}$, $i = 1,\dots,k$. Now one can choose a small open neighbourhood V of b_0 in $\mathcal{O}$ and an open neigbourhood $\tilde{V}$ of b_0 in B such that $V \subset \tilde{V} \subset \tau^{-1}(W)$ and there exist smooth $\tilde{f}_1,\dots,\tilde{f}_k : \tilde{V} \to \mathbb{R}$ extending the functions $f_i|_V$ i.e. $\tilde{f}_i|_V = f_i|_V$, $i = 1,\dots,k$. To finish the proof of case (i), it suffices to define $\tilde{F} : \tilde{V} \to E$ by

$$\tilde{F}(b) = \sum_{i=1}^k \tilde{f}_i(b) v_i|_{\tau(b)}, \quad \forall b \in \tilde{V}.$$

(ii) The fact that $t \mapsto \tilde{F}(b_0 + tv)$ is a map into a fixed vector space $E|_{F(b_0)}$ is clear since $\tilde{F}(b_0 + tv) \in E|_{\eta(\tilde{F}(b_0+tv))} = E|_{\tau(b_0+tv)} = E|_{\tau(b_0)}$. Since $F|_V = \tilde{F}|_V$ and $\nu|_{b_0}(v) \in T|_{b_0}V$, we have $F_*\nu|_{b_0}(v) = \tilde{F}_*\nu|_{b_0}(v)$. Also, $t \mapsto b_0 + tv$ is a curve in $E|_{\tau(b_0)}$, and hence in E, whose tangent vector at $t = 0$ is exactly $\nu|_{b_0}(v)$. Hence

$$\nu|_{F(b_0)}(\nu|_{b_0}(v)F) = F_*\nu|_{b_0}(v) = \tilde{F}_*\nu|_{b_0}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \tilde{F}(b_0 + tv).$$

Here on the rightmost side, the derivative $=: T$ is still viewed as a tangent vector of E at $\tilde{F}(b_0)$ i.e. $t \mapsto \tilde{F}(b_0 + tv)$ is thought of as a map into E. On the other hand, if one views $t \mapsto \tilde{F}(b_0 + tv)$ as a map into a fixed linear space $E|_{\tau(b_0)}$, its derivative $=: D$ at $t = 0$, as the usual derivative of vector valued maps, is just $D = \nu|_{F(b_0)}^{-1}(T)$. In the statement, it is exactly D whose expression we wrote as $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \tilde{F}(b_0 + tv)$. This completes the proof. $\square$

Remark 3.21 The advantage of the formula in case (ii) of the above lemma is that it simplifies in many cases the computations of τ-vertical derivatives because $t \mapsto \tilde{F}(b_0 + tv)$ is a map from a real interval into a *fixed* vector space $E|_{F(b_0)}$ and hence we may use certain computational tools (e.g. Leibniz rule) coming from the ordinary vector calculus.

Proposition 3.22 Let $\mathcal{O} \subset E^* \otimes \hat{E}$ be an immersed submanifold and $\overline{T} = (T, \hat{T}), \overline{S} = (S, \hat{S}) \in C^\infty(\overline{\eta}|_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$, be such that, for all $q = (x, \hat{x}; A) \in \mathcal{O}$,

$$\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q, \mathscr{L}_{\overline{\nabla}}(\overline{S}(q))|_q \in T|_q\mathcal{O}.$$

Then, for every $q = (x, \hat{x}; A) \in \mathcal{O}$ one has

$$\begin{aligned}[\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot)), \mathscr{L}_{\overline{\nabla}}(\overline{S}(\cdot))]|_q =& \mathscr{L}_{\overline{\nabla}}\Big(\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q \circ \overline{S} - \mathscr{L}_{\overline{\nabla}}(\overline{S}(q))|_q \circ \overline{T}\Big)\Big|_q \\ &+ \nu\big(AR^\nabla(T(q), S(q)) - R^{\hat{\nabla}}(\hat{T}(q), \hat{S}(q))A\big)\big|_q,\end{aligned} \tag{5}$$

with both sides tangent to $\mathcal{O}$.

Before the proof, we make a remark.

Remark 3.23 We discuss here the notation of $\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q \circ \overline{S}$. Since $\overline{S} : \mathcal{O} \to T(M \times \hat{M})$, we may think that $\overline{S} : C^\infty(M \times \hat{M}) \to C^\infty(\mathcal{O})$ whose value at $h \in C^\infty(M \times \hat{M})$ is $\overline{S}(\cdot)h : \mathcal{O} \to \mathbb{R}$; $q \mapsto \overline{S}(q)h$. Also, since for $q \in \mathcal{O}$, $\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q \in T|_q\mathcal{O}$, we may think this vector as a map $C^\infty(\mathcal{O}) \to \mathbb{R}$. In this way, the composition $\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q \circ \overline{S}$ is a $\mathbb{R}$-linear map $C^\infty(M \times \hat{M}) \to \mathbb{R}$, but is not a derivation (i.e. a tangent vector of $M \times \hat{M}$) in general.

But the difference $\overline{D} := \mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q \circ \overline{S} - \mathscr{L}_{\overline{\nabla}}(\overline{S}(q))|_q \circ \overline{T} : C^\infty(M \times \hat{M})$ *is* a derivation. Indeed, let $f, h \in C^\infty(M \times \hat{M})$ and compute (the details being evident)

$$\begin{aligned}&\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q(\overline{S}(\cdot)(fh)) = \mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q(h\overline{S}(\cdot)f + f\overline{S}(\cdot)h) \\ =& h\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q(\overline{S}(\cdot)f) + (\overline{T}(q)h)(\overline{S}(q)f) + f\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q(\overline{S}(\cdot)h) + (\overline{T}(q)f)(\overline{S}(q)h).\end{aligned}$$

Thus

$$\begin{aligned}\overline{D}(fh) =& h\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q(\overline{S}(\cdot)f) - h\mathscr{L}_{\overline{\nabla}}(\overline{S}(q))|_q(\overline{T}(\cdot)f) \\ &+ f\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q(\overline{S}(\cdot)h) - f\mathscr{L}_{\overline{\nabla}}(\overline{S}(q))|_q(\overline{T}(\cdot)h) \\ =& h\overline{D}f + f\overline{D}h\end{aligned}$$

which proves that indeed $\overline{D} \in T|_{(x,\hat{x})}(M \times \hat{M})$ where $q = (x, \hat{x}; A)$.

Proof. We begin with an assumption that $\mathcal{O}$ is an open set in $E^*\otimes\hat{E}$. Let $f\in C^\infty(\mathcal{O})$. By using Proposition 3.7 to express $\mathscr{L}_{\overline{\nabla}}$ and by using the definition of ν, one obtains (for clarity, we often write $q=A$ and $\overline{T}(\tilde{A})=\overline{T}\circ\tilde{A}$ etc.)

$$
\begin{aligned}
&\mathscr{L}_{\overline{\nabla}}(\overline{T}(A))|_q\big(\mathscr{L}_{\overline{\nabla}}(\overline{S}(\cdot)(f))\big)\\
=&\overline{T}(A)(\mathscr{L}_{\overline{\nabla}}(\overline{S}(\tilde{A}))|_{\tilde{A}}(f))-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\mathscr{L}_{\overline{\nabla}}(\overline{S}(A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}))|_{A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}}(f)\\
=&\overline{T}(A)\big(\overline{S}(\tilde{A})(f(\tilde{A}))-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 f(\tilde{A}+t\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A})\big)\\
&-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{S}(A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A})(f(\tilde{A}+t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))\\
&+\frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}+s\overline{\nabla}_{\overline{S}(A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A})}(\tilde{A}+t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big)\big).
\end{aligned}
$$

At this point we use the fact that $\overline{\nabla}\tilde{A}|_{(x,\hat{x})}=0$ and the fact that $\frac{\partial}{\partial t}$ and $\overline{T}(\tilde{A})$ commute (as the obvious vector fields on $M\times\hat{M}\times\mathbb{R}$ with points $(x,\hat{x},t)$) to write the last expression in the form

$$
\begin{aligned}
&\overline{T}(A)(\overline{S}(\tilde{A})(f(\tilde{A})))-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{T}(A)(f(\tilde{A}+t\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A}))-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\overline{S}(A)(f(\tilde{A}+t\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))\\
&+\frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A+st\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big)\big).
\end{aligned}
$$

By interchanging the roles of $\overline{T}$ and $\overline{S}$ and using the definition of commutator of vector fields, we get from this

$$
\begin{aligned}
&[\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot)),\mathscr{L}_{\overline{\nabla}}(\overline{S}(\cdot))]|_q f\\
=&[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_{(x,\hat{x})}(f(\tilde{A}))+\frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A+st\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A})\big)\big)\\
&-\frac{\partial^2}{\partial t\partial s}\big|_0 f\big(A+st\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A})\big)\big)\\
=&[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_{(x,\hat{x})}(f(\tilde{A}))+\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\nu(t\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))|_q f\\
&-\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\nu(t\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A}))|_q f\\
=&[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_{(x,\hat{x})}(f(\tilde{A}))+\nu(\overline{\nabla}_{\overline{S}(A)}(\overline{\nabla}_{\overline{T}(\tilde{A})}\tilde{A}))|_q(f)-\nu(\overline{\nabla}_{\overline{T}(A)}(\overline{\nabla}_{\overline{S}(\tilde{A})}\tilde{A}))|_q f\\
=&(\tilde{A}_*[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_{(x,\hat{x})})f-\nu([\overline{\nabla}_{\overline{T}(\tilde{A})},\overline{\nabla}_{\overline{S}(\tilde{A})}]\tilde{A}))|_q f.
\end{aligned}
$$

Since $\overline{\nabla}_{[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]}\tilde{A}|_{(x,\hat{x})}=0$ and $\tilde{A}|_{(x,\hat{x})}=A=q$, we have

$$(\tilde{A}_*[\overline{T}(\tilde{A}),\overline{S}(\tilde{A})]|_{(x,\hat{x})})f=\mathscr{L}_{\overline{\nabla}}([\overline{T}(\tilde{A}),\overline{S}(\tilde{A})])|_q f,$$

and, by using also Lemma 3.19,

$$([\overline{\nabla}_{\overline{T}(\tilde{A})},\overline{\nabla}_{\overline{S}(\tilde{A})}]\tilde{A})|_{(x,\hat{x})}=R^{\overline{\nabla}}(\overline{T}(q),\overline{S}(q))A=-AR^{\nabla}(T(q),S(q))+R^{\hat{\nabla}}(\hat{T}(q),\hat{S}(q))A.$$

We have thus shown that

$$[\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot)),\mathscr{L}_{\overline{\nabla}}(\overline{S}(\cdot))]|_q=\mathscr{L}_{\overline{\nabla}}([\overline{T}(\tilde{A}),\overline{S}(\tilde{A})])|_q+\nu(AR^{\nabla}(T(q),S(q))-R^{\hat{\nabla}}(\hat{T}(q),\hat{S}(q)))|_q$$

Finally, let $h \in C^\infty(M \times \hat{M})$. Then, using again that $\tilde{A}|_{(x,\hat{x})} = A$, $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$ and the definition of $\mathscr{L}_{\overline{\nabla}}$ (see also Remark 3.23 above),

$$\overline{T}(A)(\overline{S}(\tilde{A})h) = (\tilde{A}_*\overline{T}(A))(\overline{S}(\cdot)h) = \mathscr{L}_{\overline{\nabla}}(\overline{T}(A))|_q(\overline{S}(\cdot)h) = (\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q \circ \overline{S})h$$

and so

$$[\overline{T}(\tilde{A}), \overline{S}(\tilde{A})]|_{(x,\hat{x})} = \mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q \circ \overline{S} - \mathscr{L}_{\overline{\nabla}}(\overline{S}(q))|_q \circ \overline{T}. \tag{6}$$

This completes the proof in the case where $\mathcal{O} \subset E^* \otimes \hat{E}$ is open.

Assume then that $\mathcal{O}$ is an immersed submanifold of $E^* \otimes \hat{E}$ and $\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot))$, $\mathscr{L}_{\overline{\nabla}}(\overline{S}(\cdot))$ are tangent to it. By Lemma 3.20 we may choose an open neighbourhood V of q in $\mathcal{O}$, an open neighbourhood $\tilde{V}$ of q in $E^* \otimes \hat{E}$ and $\tilde{\overline{T}}, \tilde{\overline{S}} \in C^\infty(\overline{\eta}|_V, \pi_{T(M\times\hat{M})})$ such that $\tilde{\overline{T}}|_V = \overline{T}|_V$, $\tilde{\overline{S}}|_V = \overline{S}|_V$ (take in the lemma $\eta = \overline{\eta}|_{\mathcal{O}}$, $\tau = \pi_{T(M\times\hat{M})}$, $F = \overline{T}$ or $\overline{S}$ and $b_0 = q$).

Then since $\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(\cdot))|_V = \mathscr{L}_{\overline{\nabla}}(\overline{T})|_V$, $\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{S}}(\cdot))|_V = \mathscr{L}_{\overline{\nabla}}(\overline{S})|_V$ we may compute, by using what was already proved,

$$\begin{aligned}
&[\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot)), \mathscr{L}_{\overline{\nabla}}(\overline{S})]|_q = [\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(\cdot))|_V, \mathscr{L}_{\overline{\nabla}}(\tilde{\overline{S}})|_V]|_q = [\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(\cdot)), \mathscr{L}_{\overline{\nabla}}(\tilde{\overline{S}})]|_q \\
=&\mathscr{L}_{\overline{\nabla}}\Big(\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(q))|_q \circ \tilde{\overline{S}} - \mathscr{L}_{\overline{\nabla}}(\tilde{\overline{S}}(q))|_q \circ \tilde{\overline{T}}\Big)\Big|_q \\
&+ \nu(AR^{\nabla}(T(q), S(q)) + R^{\hat{\nabla}}(\hat{T}(q), \hat{S}(q))A)|_q,
\end{aligned}$$

since $\tilde{\overline{T}}(q) = \overline{T}(q) = (T(q), \hat{T}(q))$ and $\tilde{\overline{S}}(q) = \overline{S}(q) = (S(q), \hat{S}(q))$.

Finally, letting $h \in C^\infty(M \times \hat{M})$ we easily see that

$$(\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(q))|_q \circ \tilde{\overline{S}})h = \mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q(\tilde{\overline{S}}h) = \mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q(\overline{S}h)$$

where at the last equality, we noticed that $(\tilde{\overline{S}}(\cdot)h)|_V = (\overline{S}(\cdot)h)|_V$. This completes the proof. □

Now we give a proof Proposition 3.9.

Proof. (of Proposition 3.9)
Define $\overline{T}(q') = (X, \hat{X})$, $\overline{S}(q') = (Y, \hat{Y})$. Then $\overline{T}(\tilde{A}|_{(x',\hat{x}')}) = (X|_{x'}, \hat{X}|_{\hat{x}'})$, $\overline{S}(\tilde{A}|_{(x',\hat{x}')}) = (Y|_{x'}, \hat{Y}|_{\hat{x}'})$ for all $q' = (x', \hat{x}'; A')$ near q and therefore (see Eq. (6) above) one gets $[\overline{T}(\tilde{A}), \overline{S}(\tilde{A})]|_{(x,\hat{x})} = ([X,Y]|_x, [\hat{X}, \hat{Y}]|_{\hat{x}})$. □

Proposition 3.24 Let $\mathcal{O} \subset E^* \otimes \hat{E}$ be an immersed submanifold, $\overline{T} = (T, \hat{T}) \in C^\infty(\overline{\eta}|_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$, $U \in C^\infty(\overline{\eta}|_{\mathcal{O}}, \overline{\eta})$ be such that, for all $q = (x, \hat{x}; A) \in \mathcal{O}$,

$$\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q \in T|_q\mathcal{O}, \quad \nu(U(q))|_q \in T|_q\mathcal{O}.$$

Then

$$[\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot)), \nu(U(\cdot))]|_q = -\mathscr{L}_{\overline{\nabla}}(\nu(U(q))|_q\overline{T})|_q + \nu(\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_qU)|_q,$$

with both sides tangent to $\mathcal{O}$.

Remark 3.25 (i) To see that the expression $\nu(U(q))|_q\overline{T}$ makes sense according to Definition 2.1, take there $\rho := \pi_{T(M\times\hat{M})} : T(M\times\hat{M}) \to M\times\hat{M}$, $F := \overline{T} : \mathcal{O} \to T(M\times\hat{M})$, $\tau := \overline{\eta}|_{\mathcal{O}} : \mathcal{O} \to M \times \hat{M}$ and $\mathcal{V} := \nu(U(q))|_q$.

(ii) The expression $\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q U$ makes sense, in view of Definition 3.16, since $\overline{T}(q) \in T|_{(x,\hat{x})}(M \times \hat{M})$ and $U : \mathcal{O} \to E^* \otimes \hat{E} = \mathcal{T}_0^1(E) \otimes \mathcal{T}_1^0(\hat{E})$ (one takes this for S in the definition).

Proof. We will deal first with the case where $\mathcal{O}$ is an open subset of $E^* \otimes \hat{E}$. Take a local $\overline{\eta}$-section $\tilde{A}$ around $(x,\hat{x})$ such that $\tilde{A}|_{(x,\hat{x})} = A$, $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$; see Lemma 3.18. For clarity, we write often $q = A$ and $f(\tilde{A}) = f \circ \tilde{A}$ etc.

Let $f \in C^\infty(E^* \otimes \hat{E})$. Then $\mathscr{L}_{\overline{\nabla}}(\overline{T}(A))|_q\big(\nu(U(\cdot))(f)\big)$ is equal to

$$\overline{T}(A)\big(\nu(U(\tilde{A}))\big|_{\tilde{A}}(f)\big) - \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \nu(U(A + t\overline{\nabla}_{\overline{T}(A)}\tilde{A}))\big|_{A+t\overline{\nabla}_{\overline{T}(A)}\tilde{A}}(f),$$

which is equal to $\overline{T}(A)\big(\nu(U(\tilde{A}))\big|_{\tilde{A}}(f)\big)$ once we recall that $\overline{\nabla}_{\overline{T}(A)}\tilde{A} = 0$. In addition, one has

$$\begin{aligned}
&\nu(U(A))|_q\big(\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot))(f)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \mathscr{L}_{\overline{\nabla}}(\overline{T}(A + tU(A))\big|_{A+tU(A)}(f) \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \overline{T}(A + tU(A))\big(f(\tilde{A} + tU(\tilde{A}))\big) \\
&- \frac{\partial^2}{\partial s \partial t}\Big|_0 f\big(A + tU(A) + s\overline{\nabla}_{\overline{T}(A+tU(A))}(\tilde{A} + tU(\tilde{A}))\big) \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \overline{T}(A + tU(A))\big(f(\tilde{A} + tU(\tilde{A}))\big) - \frac{\partial^2}{\partial s \partial t}\Big|_0 f\big(A + tU(A) + st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big),
\end{aligned}$$

since $\overline{\nabla}_{\overline{T}(A+tU(A))}\tilde{A} = 0$.

We next simplify the first term on the last line to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \overline{T}(A + tU(A))\big(f(\tilde{A} + tU(\tilde{A}))\big) \\
=&(\nu(U(q))|_q\overline{T})\big(f(\tilde{A})\big) + \overline{T}(A)\big(\nu(U(\tilde{A}))|_{\tilde{A}}(f)\big)
\end{aligned}$$

and then, for the second term, one obtains

$$\begin{aligned}
&\frac{\partial^2}{\partial s \partial t}\Big|_0 f\big(A + tU(A) + st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big) \\
=&\frac{\mathrm{d}}{\mathrm{d}s}\Big|_0 f_*|_q \nu\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0\big(tU(A) + st\overline{\nabla}_{\overline{T}(A+tU(A))}(U(\tilde{A}))\big)\Big)\Big|_q \\
=&\frac{\mathrm{d}}{\mathrm{d}s}\Big|_0 f_*|_q \nu\big(U(A) + s\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q \\
=&\frac{\mathrm{d}}{\mathrm{d}s}\Big|_0 \Big(f_*|_q \nu(U(A))|_q + sf_*|_q \nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)|_q\Big) \\
=&f_*\nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)|_q = \nu(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A})))|_q f.
\end{aligned}$$

Therefore one deduces

$$\begin{aligned}
&[\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot)), \nu(U(\cdot))]|_q(f) = -(\nu(U(q))|_q\overline{T})\big(f(\tilde{A})\big) + \nu(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A})))|_q f \\
=& -\tilde{A}_*(\nu(U(A))|_q\overline{T})(f) + \nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q(f) \\
=& -\mathscr{L}_{\overline{\nabla}}(\nu(U(A))|_q\overline{T})|_q(f) + \nu\big(\overline{\nabla}_{\overline{T}(A)}(U(\tilde{A}))\big)\big|_q(f),
\end{aligned}$$

where the last line follows from the definition of $\mathscr{L}_{\overline{\nabla}}$ and the fact that $\overline{\nabla}_{\nu(U(A))|_q\overline{T}}\tilde{A}=0$. Finally, Proposition 3.7 implies

$$\overline{\nabla}_{T(q)}(U(\tilde{A}))=\overline{\nabla}_{T(q)}(U(\tilde{A}))-\nu(\underbrace{\overline{\nabla}_{T(q)}\tilde{A}}_{=0})|_qU=\mathscr{L}_{\overline{\nabla}}(\overline{T}(A))|_qU.$$

Thus the claimed formula holds in the special case where $\mathcal{O}$ is an open subset of $E^*\otimes\hat{E}$.

More generally, let $\mathcal{O}\subset E^*\otimes\hat{E}$ be an immersed submanifold, and $\overline{T}=(T,\hat{T}):\mathcal{O}\to T(M\times\hat{M})=TM\times T\hat{M}$, $U:\mathcal{O}\to E^*\otimes\hat{E}$ as in the statement of this proposition.

For a fixed $q=(x,\hat{x};A)\in\mathcal{O}$, Lemma 3.20 implies the existence of a neighbourhood V of q in $\mathcal{O}$, a neighbourhood $\tilde{V}$ of q in $E^*\otimes\hat{E}$ and smooth $\tilde{\overline{T}}:\tilde{V}\to T(M\times\hat{M})$, $\tilde{U}:\tilde{V}\to E^*\otimes\hat{E}$ such that $\tilde{\overline{T}}(x,\hat{x};A)\in T|_{(x,\hat{x})}(M\times\hat{M})$, $\tilde{U}(x,\hat{x};A)\in E^*\otimes\hat{E}|_{(x,\hat{x})}$ and $\tilde{\overline{T}}|_V=\overline{T}|_V$, $\tilde{U}|_V=U|_V$. Indeed, for the case of $\tilde{T}$ (resp. $\tilde{U}$), take in take in Lemma 3.20, $\tau=\overline{\eta}$, $\eta=\pi_{T(M\times\hat{M})}$, $F=\overline{T}$, $b_0=q$ (resp. $\tau=\overline{\eta}$, $\eta=\overline{\eta}$, $F=U$, $b_0=q$).

Using that $\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot))$ and $\nu(U(\cdot))$ are tangent to $\mathcal{O}$ and are the restrictions of $\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(\cdot))$, $\mathscr{L}_{\overline{\nabla}}(\tilde{U}(\cdot))$ to $\mathcal{O}$, by the above proven case where $\mathcal{O}$ was open (now applied to $\tilde{V}$),

$$\begin{aligned}
[\mathscr{L}_{\overline{\nabla}}(\overline{T}(\cdot)),\mathscr{L}_{\overline{\nabla}}(U(\cdot))]|_q=&[\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(\cdot))|_V,\mathscr{L}_{\overline{\nabla}}(\tilde{U}(\cdot))|_V]|_q=([\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(\cdot)),\mathscr{L}_{\overline{\nabla}}(\tilde{U}(\cdot))]|_V)|_q\\
=&-\mathscr{L}_{\overline{\nabla}}\big(\nu(\tilde{U}(q))|_q\tilde{\overline{T}}\big)\big|_q+\nu\big(\mathscr{L}_{\overline{\nabla}}(\tilde{\overline{T}}(q))|_q\tilde{U}\big)\big|_q\\
=&-\mathscr{L}_{\overline{\nabla}}\big(\nu(U(q))|_q\overline{T}\big)\big|_q+\nu\big(\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_qU\big)\big|_q,
\end{aligned}$$

where the last equality follows easily from Definitions 2.1 and 3.16. □

Proposition 3.26 Let $\mathcal{O}\subset E^*\otimes\hat{E}$ be an immersed submanifold and $U,V\in C^\infty(\overline{\eta}|_{\mathcal{O}},\overline{\eta})$ be such that $\nu(U(q))|_q,\nu(V(q))|_q\in T|_q\mathcal{O}$ for all $q\in\mathcal{O}$. Then

$$[\nu(U(\cdot)),\nu(V(\cdot))]|_q=\nu\big(\nu(U(q))|_qV-\nu(V(q))|_qU\big)|_q. \tag{7}$$

Remark 3.27 Here the formula $\nu(U(q))|_qV$ makes sense in terms of Definition 2.1 by taking there $\rho:=\overline{\eta}:E^*\otimes\hat{E}\to M\times\hat{M}$, $\tau:=\overline{\eta}|_{\mathcal{O}}:\mathcal{O}\to M\times\hat{M}$, $F:=V:\mathcal{O}\to E^*\otimes\hat{E}$ and $\mathcal{V}:=\nu(U(q))|_q$. Then of course $\nu(V(q))|_qU$ is also a valid expression.

Proof. Again we begin with the case where $\mathcal{O}$ is an open subset of $E^*\otimes\hat{E}$ and write $q=(x,\hat{x};A)\in\mathcal{O}$ simply as A. Let $f\in C^\infty(E^*\otimes\hat{E})$. Then,

$$\begin{aligned}
&\nu(U(A))|_q\big(\nu(V(\cdot))(f)\big)=\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\nu(V(A+tU(A))|_{A+tU(A)}(f)\\
=&\frac{\partial^2}{\partial t\partial s}\big|_0f(A+tU(A)+sV(A+tU(A)))\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0f_*|_q\nu\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big|_0\big(tU(A)+sV(A+tU(A))\big)\Big)\Big|_q\\
=&\frac{\mathrm{d}}{\mathrm{d}s}\big|_0f_*\nu\big(U(A)+s\nu(U(A))|_qV\big)|_q\\
=&f_*\nu\big(\nu(U(A))|_qV\big)|_q=\nu(\nu(U(A))|_qV)|_qf.
\end{aligned}$$

from which the result follows in the case that $\mathcal{O}$ is an open subset of $E^*\otimes\hat{E}$.

The case where $\mathcal{O}$ is only an immersed submanifold of $E^*\otimes\hat{E}$ can be treated by using Lemma 3.20 in the same way as in the proof of Proposition 3.24. □

4 The Case of $T^*M \otimes T\hat{M}$

In this section we restrict to the case where $E = TM$, $\hat{E} = T\hat{M}$ and, of course, $\eta = \pi_{TM}$, $\hat{\eta} = \pi_{T\hat{M}}$. Also, we write $\pi_{T^*M\otimes T\hat{M}}$ for $(\pi_{TM})_{\mathcal{T}^1_0} \otimes (\pi_{T\hat{M}})_{\mathcal{T}^0_1} = \pi_{T^*M} \otimes \pi_{T\hat{M}} : T^*M \otimes T\hat{M} \to M \times \hat{M}$.

4.1 A Refined Lie Bracket Formula in $T^*M \otimes T\hat{M}$ and an Application

In this setting, since one has $\mathcal{T}^0_1(TM \times T\hat{M}) = T(M \times \hat{M})$, one may reformulate Proposition 3.22 in the following form which does not involve $\tilde{A}$.

Proposition 4.1 Let $\mathcal{O} \subset T^*M \otimes T\hat{M}$ be an immersed submanifold, $\overline{S}_1 = (S_1, \hat{S}_1), \overline{S}_2 = (S_2, \hat{S}_2) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T(M\times\hat{M})})$ with $\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q, \mathscr{L}_{\overline{\nabla}}(\overline{S}_2(q))|_q \in T|_q\mathcal{O}$ for all $q = (x, \hat{x}; A) \in \mathcal{O}$. Then, for every $q \in \mathcal{O}$, one has

$$\begin{aligned}
[\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(\cdot)), \mathscr{L}_{\overline{\nabla}}(\overline{S}_2(\cdot))]|_q =& \mathscr{L}_{\overline{\nabla}}\big(\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q\overline{S}_2 - \mathscr{L}_{\overline{\nabla}}(\overline{S}_2(q))|_q\overline{S}_1\big)\big|_q \\
&- \mathscr{L}_{\overline{\nabla}}\big(T^\nabla(S_1(q), S_2(q)), T^{\hat{\nabla}}(\hat{S}_1(q), \hat{S}_2(q))\big)|_q \\
&+ \nu\big(AR(S_1(q), S_2(q)) - \hat{R}(\hat{S}_1(q), \hat{S}_2(q))A\big)\big|_q,
\end{aligned} \tag{8}$$

with both sides tangent to $\mathcal{O}$.

Remark 4.2 Notice that now $\mathscr{L}_{\overline{\nabla}}(\overline{T}(q))|_q\overline{S}$ makes sense in this setting, i.e. when $E = TM$, $\hat{E} = T\hat{M}$, according to Definition 3.16.

Proof. We may assume that $\mathcal{O} \subset T^*M \otimes T\hat{M}$ is open, since the general case where $\mathcal{O}$ is an immersed submanifold then follows by using Lemma 3.20 as indicated in the proofs of Propositions 3.22 and 3.24.

Let $h \in C^\infty(M\times\hat{M})$ and $\tilde{A}$ be a local $\overline{\eta}$-section such that $\tilde{A}|_{(x,\hat{x})} = A$ and $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$. We saw in the proof of Proposition 3.22 that

$$\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q \circ \overline{S}_2 - \mathscr{L}_{\overline{\nabla}}(\overline{S}_2(q))|_q \circ \overline{S}_1 = [\overline{S}_1(\tilde{A}), \overline{S}_2(\tilde{A})]|_{(x,\hat{x})}.$$

But

$$[\overline{S}_1(\tilde{A}), \overline{S}_2(\tilde{A})]|_{(x,\hat{x})} = \overline{\nabla}_{\overline{S}_1(q)}(\overline{S}_2(\tilde{A})) - \overline{\nabla}_{\overline{S}_2(q)}(\overline{S}_1(\tilde{A})) - T^{\overline{\nabla}}(\overline{S}_1(q), \overline{S}_2(q)).$$

For an arbitrary $\overline{\omega} \in \Gamma(\pi_{T^*(M\times\hat{M})})$ we have

$$\begin{aligned}
\overline{\nabla}_{\overline{S}_1(q)}(\overline{S}_2(\tilde{A}))\overline{\omega} =& \overline{S}_1(q)(\overline{S}_2(\tilde{A})\overline{\omega}) - \overline{S}_2(q)\overline{\nabla}_{\overline{S}_1(q)}\overline{\omega} \\
=& (\tilde{A}_*\overline{S}_1(q))(\overline{S}_2(\cdot)\overline{\omega}) - \overline{S}_2(q)\overline{\nabla}_{\overline{S}_1(q)}\overline{\omega} \\
=& \mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q\big(\overline{S}_2(\cdot)\overline{\omega}\big) - \overline{S}_2(q)\overline{\nabla}_{\overline{S}_1(q)}\overline{\omega},
\end{aligned}$$

where at the last equality we used the definition of $\mathscr{L}_{\overline{\nabla}}$ along with the fact that $\overline{\nabla}\tilde{A}|_{(x,\hat{x})} = 0$. In view of the Definition 3.16 this implies that

$$\overline{\nabla}_{\overline{S}_1(q)}(\overline{S}_2(\tilde{A})) = \mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q\overline{S}_2.$$

Thus we have arrive at

$$\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q \circ \overline{S}_2 - \mathscr{L}_{\overline{\nabla}}(\overline{S}_2(q))|_q \circ \overline{S}_1 = \mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q \overline{S}_2 - \mathscr{L}_{\overline{\nabla}}(\overline{S}_2(q))|_q \overline{S}_1 - T^{\overline{\nabla}}(\overline{S}_1(q), \overline{S}_2(q)). \tag{9}$$

The proposition is thus proved as soon as we observe that

$$T^{\overline{\nabla}}(\overline{S}_1(q), \overline{S}_2(q)) = \big(T^{\nabla}(S_1(q), S_2(q)), T^{\hat{\nabla}}(\hat{S}_1(q), \hat{S}_2(q))\big),$$

which is obvious. □

Remark 4.3 We given an alternative argument to show Eq. (9) without resorting to an extension $\tilde{A}$ of A. First we notice that by Definition 3.16 and for any $f \in C^\infty(M \times \hat{M})$ we have

$$(\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q \circ \overline{S}_2) f = (\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q \overline{S}_2) f + \overline{S}_2(q) \overline{\nabla}_{\overline{S}_1(q)} \mathrm{d} f,$$

from which we see that (9) follows if we prove that

$$\overline{S}_1(q) \overline{\nabla}_{\overline{S}_2(q)} \mathrm{d} f - \overline{S}_2(q) \overline{\nabla}_{\overline{S}_1(q)} \mathrm{d} f = T^{\overline{\nabla}}(\overline{S}_1(q), \overline{S}_2(q)) f.$$

But this true in general: If N is a manifold with connection $\tilde{\nabla}$ in TN and if $h \in C^\infty(N)$, $X, Y \in \mathrm{VF}(N)$, then

$$\begin{aligned} & X \tilde{\nabla}_Y \mathrm{d} h - Y \tilde{\nabla}_X \mathrm{d} h = Y(X \mathrm{d} h) - (\tilde{\nabla}_Y X) \mathrm{d} h - X(Y \mathrm{d} h) + (\tilde{\nabla}_X Y) \mathrm{d} h) \\ =& ([Y, X] - \tilde{\nabla}_Y X + \tilde{\nabla}_X Y) h = -T^{\tilde{\nabla}}(Y, X) h = T^{\tilde{\nabla}}(X, Y) h. \end{aligned}$$

Hence Eq. (9) follows.

As an application of Proposition 4.1, we study briefly the integrability of a rank n subdistribution of $\mathcal{D}_{\overline{\nabla}}$ which is defined in a very natural way. Indeed, it is important to notice that if $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ and $X \in T|_x M$, then $AX \in T|_{\hat{x}} \hat{M}$ and thus one may consider vectors $\mathscr{L}_{\overline{\nabla}}(X, AX)|_q$ in $\mathcal{D}_{\overline{\nabla}}$. This calls for the following definition.

Definition 4.4 For every $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ and $X \in T|_x M$ one defines the *rolling lift* of X to $\mathcal{D}_{\overline{\nabla}}$ as

$$\mathscr{L}_{\mathrm{R}}(X)|_q := \mathscr{L}_{\overline{\nabla}}(X, AX)|_q.$$

Moreover, define a subdistribution $\mathcal{D}_{\mathrm{R}}$, called *rolling distribution*, of rank n of $\mathcal{D}_{\overline{\nabla}}$ as

$$\mathcal{D}_{\mathrm{R}}|_q := \mathscr{L}_{\mathrm{R}}(T|_x M)|_q.$$

Of course, if $X \in \mathrm{VF}(M)$, then $\mathscr{L}_{\mathrm{R}}(X) : T^*M \otimes T\hat{M} \to T(T^*M \otimes T\hat{M})$ such that $q \mapsto \mathscr{L}_{\mathrm{R}}(X)|_q$ is a smooth vector field in $T^*M \otimes T\hat{M}$.

As an application of the last proposition, we immediately get a formula for the Lie brackets of vector fields of the form $\mathscr{L}_{\mathrm{R}}(X)$. Before stating the result, we make the following definitions.

Definition 4.5 For $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$ and $X, Y \in T|_x M$ one defines the *rolling curvature* as

$$R^{\mathsf{Rol}}|_q(X, Y) := AR^{\nabla}(X, Y) - R^{\hat{\nabla}}(AX, AY)A$$

and the *rolling torsion* as

$$T^{\mathsf{Rol}}|_q(X, Y) := AT^{\nabla}(X, Y) - T^{\hat{\nabla}}(AX, AY).$$

The term "rolling" will be explained in section 4.3. With these notations, we may formulate the above mentioned Lie bracket formula as follows.

Proposition 4.6 Let $X, Y \in \mathrm{VF}(M)$ and $q = (x, \hat{x}; A) \in T^*M \otimes T\hat{M}$. Then

$$[\mathscr{L}_{\mathrm{R}}(X), \mathscr{L}_{\mathrm{R}}(Y)]|_q = \mathscr{L}_{\mathrm{R}}([X, Y])|_q + \mathscr{L}_{\overline{\nabla}}(T^{\mathsf{Rol}}(X, Y))|_q + \nu(R^{\mathsf{Rol}}(X, Y))|_q.$$

Proof. If in Proposition 4.1 one takes $\overline{S}_1(q) = (X, AX)$, $\overline{S}_2(q) = (Y, AY)$, if $q = (x, \hat{x}; A)$, one easily verifies that $\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q \overline{S}_2 = (\nabla_X Y, A\nabla_X Y)$ and $\mathscr{L}_{\overline{\nabla}}(\overline{S}_2(q))|_q \overline{S}_1 = (\nabla_Y X, A\nabla_Y X)$. But then

$$\begin{aligned}\mathscr{L}_{\overline{\nabla}}(\mathscr{L}_{\overline{\nabla}}(\overline{S}_1(q))|_q \overline{S}_2 - \mathscr{L}_{\overline{\nabla}}(\overline{S}_2(q))|_q \overline{S}_1)|_q =& \mathscr{L}_{\mathrm{R}}(\nabla_X Y - \nabla_Y X)|_q \\ =& \mathscr{L}_{\mathrm{R}}([X, Y])|_q + \mathscr{L}_{\mathrm{R}}(T^{\nabla}(X, Y))|_q\end{aligned}$$

and

$$\begin{aligned}&\mathscr{L}_{\mathrm{R}}(T^{\nabla}(X, Y))|_q - \mathscr{L}_{\overline{\nabla}}\big(T^{\nabla}(X, Y), T^{\hat{\nabla}}(AX, AY)\big)|_q \\ =& \mathscr{L}_{\overline{\nabla}}\big((T^{\nabla}(X, Y), AT^{\nabla}(X, Y)) - (T^{\nabla}(X, Y), T^{\hat{\nabla}}(AX, AY)\big)|_q \\ =& \mathscr{L}_{\overline{\nabla}}(T^{\mathsf{Rol}}(X, Y))|_q.\end{aligned}$$

From this the proposition readily follows. □

One can characterize the integrability of $\mathcal{D}_{\mathrm{R}}$ as follows (compare to Corollary 3.12).

Corollary 4.7 An orbit $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ of $T^*M \otimes T\hat{M}$ is an integral manifold of $\mathcal{D}_{\mathrm{R}}$ if and only if $R^{\mathsf{Rol}}|_q = 0$ and $T^{\mathsf{Rol}}|_q = 0$ for all $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$.

Proof. If $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ is an integral manifold of $\mathcal{D}_{\mathrm{R}}$, Proposition 4.6 immediately implies that R^{Rol} and T^{Rol} vanish at each point of $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$.

Conversely, suppose $R^{\mathsf{Rol}}|_q = 0$ and $T^{\mathsf{Rol}}|_q = 0$ for all $q \in \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$. Then Proposition 4.6 implies that $\mathcal{D}_{\mathrm{R}}|_{\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)}$ is an involutive distribution in $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ and hence there exists a maximal integral manifold N through q_0. Obviously then $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = N$. □

One can characterize integral manifolds of $\mathcal{D}_{\mathrm{R}}$ in more geometric terms, but this task is beyond the scope of this paper.

4.2 The Riemannian Case

In this section, we assume that (M, g) and $(\hat{M}, \hat{g})$ Riemannian manifolds and ∇, $\hat{\nabla}$ are their Levi-Civita connections, respectively.

The following proposition holds in this general setting, but after it, we will make more assumptions on $M, \hat{M}$.

Proposition 4.8 If M and $\hat{M}$ are simply-connected, then each $\pi_{T^*M\otimes T\hat{M}}$-fiber $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)\cap T|_x^*M\otimes T|_{\hat{x}}\hat{M}$, with $(x,\hat{x})\in M\times\hat{M}$, of any orbit $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$, $q_0=(x_0,\hat{x}_0;A_0)$, is a compact connected submanifold $T^*M\otimes T\hat{M}$.

Proof. Without loss of generality we may assume that $(x,\hat{x})=(x_0,\hat{x}_0)$. It is well known that the simply connectedness assumption implies that $H^{\nabla}|_{x_0}$ and $\hat{H}^{\hat{\nabla}}|_{\hat{x}_0}$ are respectively (closed and hence) compact connected Lie-subgroups of $\mathrm{SO}(T|_{x_0}M)$ and $\mathrm{SO}(T|_{\hat{x}_0}\hat{M})$. See Theorem 3.2.8 in [10] or Appendix 5 in [11]. Thus $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)\cap T^*M|_{x_0}\otimes T\hat{M}|_{\hat{x}_0}$ is compact and connected since it is a continuous image of the compact connected set $G:=H^{\hat{\nabla}}|_{\hat{x}_0}\times H^{\nabla}|_{x_0}$ by the (smooth) map $\psi:G\to\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$; $\phi(\hat{h},h)=\hat{h}\circ A_0\circ h^{-1}$ (see e.g. the proof of Proposition 3.13). $\square$

For the rest of the section, we assume that $M,\hat{M}$ are moreover *oriented* and have *the same dimensions, $n=\hat{n}$.* We define

$$Q(M,\hat{M}):=\{(x,\hat{x};A)\in T^*M\otimes T\hat{M}\ |\ \|AX\|_{\hat{g}}=\|X\|_g,\ \forall X\in T|_xM,\ \det A=+1\}.$$

Here $\|X\|_g=g(X,X)^{1/2}$, $\left\|\hat{X}\right\|_{\hat{g}}=\hat{g}(\hat{X},\hat{X})^{1/2}$ and notice that the determinant of $A:T|_xM\to T|_{\hat{x}}\hat{M}$ is defined since $M,\hat{M}$ are oriented (and orientation is fixed once and for all). We abbreviate Q for $Q(M,\hat{M})$ when $M,\hat{M}$ are clear from the context.

It is clear that if $\pi_Q:=\pi_{T^*M\otimes T\hat{M}}|_Q$, then $\pi_Q:Q\to M\times\hat{M}$ is a smooth bundle and that its fibers are diffeomorphic to $\mathrm{SO}(n)$, so $\dim Q=n+n+\frac{n(n-1)}{2}$. Thus the fibers $Q|_{(x,\hat{x})}$ are connected for all $(x,\hat{x})$ and hence that Q connected, if $M,\hat{M}$ are connected.

We will show that $\mathcal{D}_{\overline{\nabla}}$ restricts to a smooth distribution on Q of rank $2n$.

Lemma 4.9 For every $q\in Q$ one has $\mathcal{D}_{\overline{\nabla}}|_q\subset T|_qQ$ and $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)\subset Q$.

Proof. Let $q=(x,\hat{x};A)\in Q$ and $\overline{\gamma}=(\gamma,\hat{\gamma})$ a smooth path in $M\times\hat{M}$ with $\overline{\gamma}=(x,\hat{x})$. By Lemma 3.6 (i) one has

$$(P^{\overline{\nabla}})^t_0(\overline{\gamma})A=(P^{\hat{\nabla}})^t_0(\hat{\gamma})\circ A_0\circ(P^{\nabla})^0_t(\gamma).$$

Since $(P^{\nabla})^0_t(\gamma)$ (resp. $(P^{\hat{\nabla}})^t_0(\hat{\gamma})$) is an g-orthogonal (resp. $\hat{g}$-orhogonal) map $T|_{\gamma(t)}M\to T|_{x_0}M$ (resp. $T|_{\hat{x}_0}\hat{M}\to T|_{\hat{\gamma}(t)}\hat{M}$) and A_0 is an orthogonal map $(T|_{x_0}M,g)\to(T|_{\hat{x}_0}\hat{M},\hat{g})$, we obtain that $(P^{\overline{\nabla}})^t_0(\overline{\gamma})A$ is an orthogonal map $(T|_{\gamma(t)}M,g)\to(T|_{\hat{\gamma}(t)}\hat{M},\hat{g})$. Thus for all t one has $\det(P^{\overline{\nabla}})^t_0(\overline{\gamma})A\in\{+1,-1\}$. Since for $t=0$ this has value $+1$, because $A_0\in Q$, it is always $+1$. Thus for all t, $(P^{\overline{\nabla}})^t_0(\overline{\gamma})A\in Q|_{(\gamma(t),\hat{\gamma}(t))}$ and so

$$TQ\ni\frac{\mathrm{d}}{\mathrm{d}t}\big|_0(P^{\overline{\nabla}})^t_0(\overline{\gamma})A\in Q|_{(\gamma(t),\hat{\gamma}(t))}=\mathscr{L}_{\overline{\nabla}}(\dot{\overline{\gamma}}(0))|_q.$$

This proves that $\mathcal{D}_{\overline{\nabla}}$ is tangent to Q (i.e. $\mathcal{D}_{\overline{\nabla}}|_Q\subset TQ$) from which then $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)\subset Q$ follows as well. $\square$

As a corollary to Proposition 4.8 we have the following result.

Corollary 4.10 Assuming that M and $\hat{M}$ are simply-connected, if an orbit $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$ for $q_0\in Q$ is open in Q then $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)=Q$.

Proof. The claim follows from the fact that an open orbit $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$ has a open fiber $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0) \cap Q|_{(x_0,\hat{x}_0)}$ in $Q|_{(x_0,\hat{x}_0)}$. This fiber is also compact by what we just proved and hence $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0) \cap Q|_{(x_0,\hat{x}_0)} = Q|_{(x_0,\hat{x}_0)}$ by connectedness of $Q|_{(x_0,\hat{x}_0)}$. This clearly implies that $Q = \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$. □

Proposition 4.11 Suppose $M, \hat{M}$ are simply connected. Then $\mathcal{D}_{\overline{\nabla}}$ is completely controllable in Q if and only if

$$\mathfrak{h}^{\nabla}|_F + \mathfrak{h}^{\hat{\nabla}}|_{\hat{F}} = \mathfrak{so}(n), \tag{10}$$

for some (and hence any) orthonormal frames $F, \hat{F}$ of $M, \hat{M}$, respectively.

Proof. Let $F = (X_i), \hat{F} = (\hat{X}_i)$ by any orthonormal frames at $x_0 \in M$, $\hat{x}_0 \in \hat{M}$, resp. We may assume w.l.o.g. that they are be oriented. Define $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ by $A_0 X_i = \hat{X}_i$, $i = 1, \ldots, n$. Then

$$\mathcal{M}_{F,\hat{F}}\big(\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}_0} \circ A_0 + A_0 \circ \mathfrak{h}^{\nabla}|_{x_0}\big) = \mathfrak{h}^{\hat{\nabla}}|_{\hat{F}} + \mathfrak{h}^{\nabla}|_F,$$

We also point out that, as is very easy to see,

$$\nu|_{q_0}^{-1}(V|_{q_0}(\pi_Q)) = A_0 \circ \mathfrak{so}(T|_{x_0}M)$$

and that $\mathcal{M}_{\hat{F},F}(A_0 \circ \mathfrak{so}(T|_{x_0}M)) = \mathfrak{so}(n)$.

Suppose then that $\mathcal{D}_{\overline{\nabla}}$ is completely controllable. This implies that $V|_{q_0}(\pi_Q) \cap \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0) = V|_{q_0}(\pi_Q)$ i.e., by Proposition 3.13, that $V|_{q_0}(\pi_Q) = \nu(\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}_0} \circ A_0 + A_0 \circ h^{\nabla}|_{x_0})|_{q_0}$. This with the above formula for $\nu|_{q_0}^{-1}(V|_{q_0}(\pi_Q))$ gives the necessary condition.

Conversely, suppose that $\mathfrak{h}^{\nabla}|_F + \mathfrak{h}^{\hat{\nabla}}|_{\hat{F}} = \mathfrak{so}(n)$ for some orthonormal frames $F, \hat{F}$, which we may assume to be oriented. Defining A_0 as above and going the above argument backwards, we get that $V|_{q_0}(\pi_Q) \cap \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0) = V|_{q_0}(\pi_Q)$. But this means that the orbit $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$ is open in Q and hence Corollary 4.10 implies that $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0) = Q$. End of the proof. □

Even though the π_Q-fibers are diffeomorphic to $\mathrm{SO}(n)$, we will show below that generally there is no ($\mathrm{SO}(n)$-)principal bundle structure on π_Q which would render $\mathcal{D}_{\overline{\nabla}}$ to a principal bundle connection.

Theorem 4.12 Generically, in dimension $n \geq 3$, π_Q cannot be equipped with a principal bundle structure which leaves $\mathcal{D}_{\overline{\nabla}}$ invariant.

More precisely, suppose that $n \geq 3$ and $F, \hat{F}$ are oriented orthonormal frames of M and $\hat{M}$ at x_0 and $\hat{x}_0$, respectively, and let $H^{\nabla}|_F \subset \mathrm{SO}(n)$, $H^{\hat{\nabla}}|_{\hat{F}} \subset \mathrm{SO}(n)$ be the holonomy groups with respect to these frames. If $H^{\nabla}|_F \cap H^{\hat{\nabla}}|_{\hat{F}}$ is not a finite subgroup of $\mathrm{SO}(n)$, then there is no principal bundle structure on π_Q which leaves $\mathcal{D}_{\overline{\nabla}}$ invariant.

Especially this holds if M (resp. $\hat{M}$) has full holonomy $\mathrm{SO}(n)$ and the holonomy group of $\hat{M}$ (resp. M) is not finite.

Proof. Suppose that $\mu : G \times Q \to Q$ is a left-principal bundle structure for π_Q which leaves $\mathcal{D}_{\overline{\nabla}}$ invariant i.e. $(\mu_g)_* \mathcal{D}_{\overline{\nabla}} = \mathcal{D}_{\overline{\nabla}}$. Define $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ by $A_0 X_i = \hat{X}_i$ for $i = 1, \ldots, n$ where $F = (X_i)$, $\hat{F} = (\hat{X}_i)$.

If $B \in H^{\nabla}|_F \cap H^{\hat{\nabla}}|_{\hat{F}}$, then Proposition 3.15 implies that B commutes with every $\mathcal{M}_{\hat{F},F}(\mu(g, A_0))$, $g \in G$. But $g \mapsto \mu(g, A_0)$ must be surjective (and in fact bijective) $G \to$

$Q|_{(x_0,\hat{x}_0)}$, which implies that B must commute with every element of $\mathcal{M}_{\hat{F},F}(\mu(G\times\{A_0\})) = \mathcal{M}_{F,\hat{F}}(Q|_{(x_0,\hat{x}_0)}) = \mathrm{SO}(n)$. Therefore $H^{\nabla}|_F \cap H^{\hat{\nabla}}|_{\hat{F}} \subset Z(\mathrm{SO}(n))$, where $Z(\mathrm{SO}(n))$ is the center of $\mathrm{SO}(n)$. Since $Z(\mathrm{SO}(n))$ is a finite subgroup of $\mathrm{SO}(n)$ (the set of all diagonal matrices $\mathrm{diag}(a_1,\dots,a_n)$ such that $a_i \in \{-1,+1\}$ and $a_1a_2\cdots a_n = +1$), it follows that $H^{\nabla}|_F \cap H^{\hat{\nabla}}|_{\hat{F}}$ is a finite subgroup of $\mathrm{SO}(n)$. This establishes the claim. □

There is a complete classification of holonomy groups of Riemannian manifolds by Cartan (for symmetric spaces, see [9]) and Berger (for non-symmetric spaces, see [10]). Hence the above theorems reduce the question of complete controllability of $\mathcal{D}_{\overline{\nabla}}$ to an essentially linear algebraic problem.

For instance, in the case where both manifolds are non-symmetric, simply connected and irreducible, we get the following proposition.

Theorem 4.13 Assume that the manifolds M and $\hat{M}$ are complete non-symmetric, simply connected, irreducible and $n \neq 8$. Then $\mathcal{D}_{\overline{\nabla}}$ is completely controllable in Q if and only if either H^{∇} or $H^{\hat{\nabla}}$ is equal to $\mathrm{SO}(n)$ (w.r.t some orthonormal frames).

Proof. Suppose first that $H^{\nabla}|_F = \mathrm{SO}(n)$. Choose any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and define $\hat{F} = A_0 F$ (which is an orthonormal frame of $\hat{M}$ at $\hat{x}_0$ since $A_0 \in Q$) and compute, noticing that $\mathcal{M}_{F,\hat{F}}(A_0) = \mathrm{id}_{\mathbb{R}^n}$,

$$\pi_Q^{-1}(x_0,\hat{x}_0) \cap \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0) \cong H^{\hat{\nabla}}|_{\hat{F}} H^{\nabla}|_F = H^{\hat{\nabla}}|_{\hat{F}}\mathrm{SO}(n) = \mathrm{SO}(n),$$

where the first diffeomorphism comes from Proposition 3.13. But the π_Q-fibers of Q are diffeomorphic (through the frames $F, \hat{F}$) to $\mathrm{SO}(n)$ and hence $\pi_Q^{-1}(x_0,\hat{x}_0) \cap \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0) = \pi_Q^{-1}(x_0,\hat{x}_0)$. By connectedness of $M, \hat{M}$ it follows that $Q = \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$.

Assume now that both holonomy groups are different from $\mathrm{SO}(n)$. We also remark that if one holonomy group is included in the other one, then complete controllability cannot hold according to Proposition 3.13 (choosing A appropriately). Using Berger's list, see [10], and taking into account that

$$\mathrm{Sp}(m) \subset \mathrm{SU}(2m) \subset \mathrm{U}(2m) \subset \mathrm{SO}(4m)$$

where $n = 4m$, it only remains to study the following case: $n = 4m$ with $m \geq 2$, one group is equal to $U(2m)$ and the other one to $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$. Recall that

$$\dim\Big(\underbrace{U(2m)\big(\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)\big)}_{U(2m)\cdot\mathrm{Sp}(1)}\Big) \leq \dim U(2m) + \dim \mathrm{Sp}(1) = 4m^2 + 3.$$

On the other hand $\dim \mathrm{SO}(4m) = 8m^2 - 2m$ which is always strictly larger than $4m^2+3$ for all $m \geq 2$.

□

We show that $\mathcal{D}_{\overline{\nabla}}$ is not completely controllable if (M,g) and $(\hat{M},\hat{g})$ are Riemannian products.

Proposition 4.14 Suppose that (M,g) and $(\hat{M},\hat{g})$ are equal to the Riemannian products $(M_1 \times M_2, g_1 \oplus g_2)$ and $(\hat{M}_1 \times \hat{M}_2, \hat{g}_1 \oplus \hat{g}_2)$, with $n_i = \dim M_i \geq 1$, $\hat{n}_i = \dim \hat{M}_i \geq 1$, $i = 1,2$ respectively. Then, $\mathcal{D}_{\overline{\nabla}}$ is not completely controllable in Q.

Proof. We need to show that, under the assumptions, there exists $q_0 = (x_0; \hat{x}_0; A_0) \in Q$ so that the orbit $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$ is a proper subset of Q. Notice that $n = n_1 + n_2 = \hat{n}_1 + \hat{n}_2$.

Fix $x = (x_1, x_2) \in M_1 \times M_2 = M$ and $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \hat{M}_1 \times \hat{M}_2 = \hat{M}$. Let $F_1 = (X_1^1, \dots, X_{n_1}^1)$ and $F_2 = (X_1^2, \dots, X_{n_2}^2)$ be an oriented orthonormal basis of $T|_{x_1}M_1$ and $T|_{x_2}M_2$ respectively and, similarly, let $\hat{F}_1 = (\hat{X}_1^1, \dots, \hat{X}_{\hat{n}_1}^1)$ and $\hat{F}_2 = (\hat{X}_1^2, \dots, \hat{X}_{\hat{n}_2}^2)$ be an oriented orthonormal basis of $T|_{\hat{x}_1}\hat{M}_1$ and $T|_{\hat{x}_2}\hat{M}_2$ respectively. Then $F = (F_1, F_2)$ and $\hat{F} = (\hat{F}_1, \hat{F}_2)$ are oriented orthonormal basis of of M and $\hat{M}$ at x and $\hat{x}$ respectively.

Writing $\mathfrak{h}|_F, \mathfrak{h}_1|_{F_1}, \mathfrak{h}_2|_{F_2}$ (resp. $\hat{\mathfrak{h}}|_{\hat{F}}, \hat{\mathfrak{h}}_1|_{\hat{F}_1}, \hat{\mathfrak{h}}_2|_{\hat{F}_2}$) for the holonomy Lie algebras of M, M_1, M_2 (resp. $\hat{M}, \hat{M}_1, \hat{M}_2$) w.r.t to the above frames, one has the direct sum splittings

$$\begin{aligned}
\mathfrak{h}|_F &= \mathfrak{h}_1|_{F_1} \oplus \mathfrak{h}_2|_{F_2} \subset \mathfrak{so}(n_1) \oplus \mathfrak{so}(n_2), \\
\hat{\mathfrak{h}}|_{\hat{F}} &= \hat{\mathfrak{h}}_1|_{\hat{F}_1} \oplus \hat{\mathfrak{h}}_2|_{\hat{F}_2} \subset \mathfrak{so}(\hat{n}_1) \oplus \mathfrak{so}(\hat{n}_2).
\end{aligned}$$

Without loss of generality, we assume that $\hat{n}_1 \geq n_1$.

Define the linear map $A : T|_x M \to T|_{\hat{x}}\hat{M}$ by

$$A(X_j^1) = \hat{X}_j^1, \quad j = 1, \dots, n_1, \quad A(X_j^2) = \hat{X}_{n_1+j}^1, \quad j = 1, \dots, \hat{n}_1 - n_1,$$

and

$$A(X_j^2) = \hat{X}_{j-(\hat{n}_1-n_1)}^2, \quad j = \hat{n}_1 - n_1 + 1, \dots, n_2.$$

Thus, we have $\mathcal{M}_{F,\hat{F}}(A) = \mathrm{id}_{\mathbb{R}^n}$, so $q = (x, \hat{x}; A) \in Q$, and

$$\begin{aligned}
\mathcal{M}_{F,\hat{F}}\big(\mathfrak{h}^{\hat{\nabla}}|_{\hat{x}} \circ A + A \circ \mathfrak{h}^{\nabla}|_x\big) =&\hat{\mathfrak{h}}|_{\hat{F}} \circ \mathcal{M}_{F,\hat{F}}(A) + \mathcal{M}_{F,\hat{F}}(A) \circ \mathfrak{h}|_F \\
=&\mathfrak{h}_1|_{\hat{F}_1} \oplus \mathfrak{h}_2|_{\hat{F}_2} + \hat{\mathfrak{h}}_1|_{F_1} \oplus \hat{\mathfrak{h}}_2|_{F_2}.
\end{aligned}$$

The latter linear vector space is necessarily a proper subset of $\mathfrak{so}(n)$. For example, if E_{ij} is the $n \times n$-matrix with 1 at the i-th row, j-th column and zero otherwise, then the above linear space does not contain $E_{n1} - E_{1n} \in \mathfrak{so}(n)$.

As mentioned in the proof of Proposition 4.11, one has $V|_q(\pi_Q) = \nu(A\mathfrak{so}(T|_x M))|_q$. Since $\mathcal{M}_{\hat{F},F}(A\mathfrak{so}(T|_x M)) = \mathfrak{so}(n)$, Proposition 3.13 implies that the tangent space $V|_q(\pi_Q) \cap T|_q\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ to the fiber of the orbit over $(x, \hat{x})$ has dimension (which is $\dim(\mathfrak{h}_1|_{\hat{F}_1} \oplus \mathfrak{h}_2|_{\hat{F}_2} + \hat{\mathfrak{h}}_1|_{F_1} \oplus \hat{\mathfrak{h}}_2|_{F_2})$) strictly lower than the fiber of Q over $(x, \hat{x})$. Thus $\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q)$ is not open in Q and therefore $\mathcal{D}_{\overline{\nabla}}$ cannot be completely controllable. □

Corollary 4.15 Suppose that both (M, g) and $(\hat{M}, \hat{g})$ are complete simply connected reducible Riemannian manifolds of dimension $n \geq 2$. Then $\mathcal{D}_{\overline{\nabla}}$ is not completely controllable.

Proof. Reducible complete simply connected Riemannian manifolds of dimension $n \geq 2$ are Riemannian products of manifolds of lower dimension ≥ 1 by the theorem of de Rham. Hence the previous proposition establishes the claim. □

4.3 Relation to the Rolling Problem

In this section we introduce the model of rolling of two Riemannian manifolds and relate it to the subdistribution $\mathcal{D}_{\mathrm{R}}$ of $\mathcal{D}_{\overline{\nabla}}$ as given in Definition 4.4. We begin with the classical model (see also [2, 4, 7, 8, 13, 15]).

Suppose $M, \hat{M} \subset \mathbb{R}^3$ are embedded oriented surfaces and make M roll agains $\hat{M}$ along some prescribed curve in M. Let $N, \hat{N}$ be the unit normal vector fields on $M, \hat{M}$, respectively. First, to make M in contact with $\hat{M}$ at the respective chosen *contact points* $x \in M$, $\hat{x} \in \hat{M}$, we need to bring the point x to the point $\hat{x}$ with an Euclidean (rigid) motion in such a way that the tangent spaces $T|_x M$, $\hat{T}|_{\hat{x}}\hat{M}$ after this motion coincide. This is equivalent to saying that normals $N|_x$ and $\hat{N}|_{\hat{x}}$ coincide after the motion.

So the Euclidean motion $(U, a) \in \mathrm{SE}(3)$, where $\mathrm{SE}(3)$ is $\mathrm{SO}(3) \times \mathbb{R}^3$ as a set, has to satisfy

$$UN|_x = \hat{N}|_{\hat{x}}, \quad a = x - \hat{x}.$$

(We think of all tangent vectors of $\mathbb{R}^3$ to be vectors based at origin i.e. $T|_u\mathbb{R}^3$ is identified with $\mathbb{R}^3$ for all $u \in \mathbb{R}^3$.) After the rigid motion determined by $x \in M$ and $(U, a) = (U, x - \hat{x})$, the surface M transforms to $U(M - x) + \hat{x}$ which satisfies $T|_{\hat{x}}(U(M - x) + \hat{x}) = T|_{\hat{x}}\hat{M}$.

Hence the set of all *contact configurations* of M and $\hat{M}$ is determined by data $(x, \hat{x}, U) \in M \times \hat{M} \times \mathrm{SO}(3)$ such that $UN|_x = \hat{N}|_{\hat{x}}$ and they constitute a set $Q_{\mathbb{R}^3} = Q_{\mathbb{R}^3}(M, \hat{M})$.

Definition 4.16 For two surfaces $M, \hat{M} \subset \mathbb{R}^3$, the set of admissible *contact configurations* is

$$Q_{\mathbb{R}^3}(M, \hat{M}) := \{(x, \hat{x}, U) \in M \times \hat{M} \times \mathrm{SO}(3) \mid UN|_x = \hat{N}|_{\hat{x}}\}.$$

Notice that a rotation $U \in \mathrm{SO}(3)$ such that $UN|_x = \hat{N}|_{\hat{x}}$ uniquely determines an orthonormal map $A_U : T|_x M \to T|_{\hat{x}}\hat{M}$ with determinant $= +1$ by $A_U := U|_{T|_x M}$. Here the tangent spaces $T|_x M$ and $T|_{\hat{x}}\hat{M}$ have the orientations determined by $N|_x, \hat{N}|_{\hat{x}}$, respectively. Conversely, an orthonormal map $A : T|_x M \to T|_{\hat{x}}\hat{M}$ of determinant $+1$ uniquely determines a rotation $U = U_A \in \mathrm{SO}(3)$ by imposing that $UN|_x = \hat{N}|_{\hat{x}}$ and $U|_{T|_x M} = A$. Hence $Q_{\mathbb{R}^3}$ is in a natural bijection with the set $Q = Q(M, \hat{M})$ defined in the beginning of the section 4.2, when M, $\hat{M}$ have the Riemannian metrics induced from $\mathbb{R}^3$. This proves the following.

Lemma 4.17 The map

$$Q_{\mathbb{R}^3}(M, \hat{M}) \to Q(M, \hat{M}); \quad (x, \hat{x}; U) \mapsto (x, \hat{x}; U|_{T|_x M})$$

is a bijection.

We see from the above discussion that for an element $q = (x, \hat{x}; A) \in Q$, the linear map A represent *relative orientation* (about normals $N, \hat{N}$) of tangent spaces of $T|_x M, T|_{\hat{x}}\hat{M}$ in contact.

We introduce next the dynamics of rolling. For any $u \in \mathbb{R}^3$ we define

$$J_u \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3); \quad J_u(v) := u \times v,$$

where $\times$ is the cross product. If u is a unit vector, then clearly $J_u \in \mathfrak{so}(3)$ and it is the infinitesimal rotation of unit speed about the axis u.

Definition 4.18 Given a curve $\gamma : [0, 1] \to M$ s.t. $\gamma(0) = x_0$, we say that M rolls against $\hat{M}$ *without spinning* and *without slipping* along γ, if there is a curve $(U(t), \hat{\gamma}(t)) \in \mathrm{SO}(3) \times \hat{M}$ such that the following conditions hold for all $t \in [0, 1]$:

(i) Contact: $U(t)N|_{\gamma(t)} = \hat{N}|_{\hat{\gamma}(t)}$;

(ii) No-slipping: $U(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$;

(iii) No-spinning: $\dot{U}(t)U(t)^{-1} \in \mathfrak{so}(3)$ is orthogonal to $J_{\hat{N}|_{\hat{\gamma}(t)}} \in \mathfrak{so}(3)$.

Recall that in $\mathfrak{so}(3)$ one defines the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{so}(3)}$ by $\langle V, W \rangle_{\mathfrak{so}(3)} := -\mathrm{tr}(VW)$, $V, W \in \mathfrak{so}(3)$.

Remark 4.19 Condition (iii) means that the rotation velocity $\dot{U}(t)U(t)^{-1}$ has zero component in $J_{\hat{N}|_{\hat{\gamma}(t)}}$-direction. Since this component exactly measures the instantaneous rotation of M about the normal $\hat{N}|_{\hat{\gamma}}(t)$, i.e. spinning, the name of this condition is motivated.

Lemma 4.20 The condition (iii) is equivalent to the following: For all $t \in [0, 1]$,

(iii)' $\dot{U}(t)U(t)^{-1} = J_{\hat{W}(t)}$, where $\hat{W}(t) \in T|_{\hat{\gamma}(t)}\hat{M}$.

Proof. Indeed, if $x, y, z \in \mathbb{R}^3$ are orthonormal and $a \in \mathfrak{so}(3)$, then $\langle J_z, a \rangle_{\mathfrak{so}(3)}$=0 iff $a = \alpha J_x + \beta J_y = J_{\alpha x + \beta y}$. □

We will interpret these conditions (i)-(iii) intrinsically i.e. in terms of the Riemannian geometry of M and $\hat{M}$ only. Since curves $(\gamma(t), \hat{\gamma}(t); U(t)) \in Q_{\mathbb{R}^3}$ are in bijection with curves $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in Q, as explained earlier, through condition (i) and since the no-slipping condition (ii) just means, by this correspondence, $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$, which is intrinsic, we see that it is enough to interpret (iii) intrinsically.

Lemma 4.21 Let $(\gamma(t), \hat{\gamma}(t), U(t))$ be a curve in $Q_{\mathbb{R}^3}$ that satisfies contact condition (i). Then it satisfies no-spinning condition (iii) if and only if for any vector field $X(t)$ along γ which is tangent to M, one has

$$\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\big(U(\cdot)X(\cdot)\big) = U(t)\nabla_{\dot{\gamma}(t)}X(\cdot). \tag{11}$$

Proof. Let $X(t)$ be a vector field in $\mathbb{R}^3$ along γ tangent to M for all t. Then is a vector field $\hat{X}(t) := U(t)X(t)$ is tangent to $\hat{M}$ (condition (i)) along $\hat{\gamma}$ and

$$\dot{\hat{X}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(U(t)X(t)) = \dot{U}(t)X(t) + U(t)\dot{X}(t).$$

Writing $\nabla, \hat{\nabla}$ for the Levi-Civita connections of $M, \hat{M}$, $\langle \cdot, \cdot \rangle$ for the inner product in $\mathbb{R}^3$ and $N(t) = N|_{\gamma(t)}$, $\hat{N}(t) = \hat{N}|_{\hat{\gamma}(t)}$, we have

$$\begin{aligned} \dot{X}(t) &= \nabla_{\dot{\gamma}(t)}X + \left\langle \dot{X}(t), N(t) \right\rangle N(t) \\ \dot{\hat{X}}(t) &= \hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{X} + \left\langle \dot{\hat{X}}(t), \hat{N}(t) \right\rangle \hat{N}(t). \end{aligned}$$

Substituting these into the above equation, we obtain

$$\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{X} + \left\langle \dot{\hat{X}}(t), \hat{N}(t) \right\rangle \hat{N}(t) = \dot{U}(t)U(t)^{-1}\hat{X}(t) + U(t)\nabla_{\dot{\gamma}(t)}X + \left\langle \dot{X}(t), N(t) \right\rangle \hat{N}(t).$$

But

$$\left\langle \dot{\hat{X}}(t), \hat{N}(t) \right\rangle = \left\langle \dot{U}(t)X(t) + U(t)\dot{X}(t), \hat{N}(t) \right\rangle = \left\langle \dot{U}(t)X(t), \hat{N}(t) \right\rangle + \left\langle \dot{X}(t), N(t) \right\rangle,$$

and so

$$\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{X} - U(t)\nabla_{\dot{\gamma}(t)}X = \dot{U}(t)U(t)^{-1}\hat{X}(t) - \left\langle \dot{U}(t)U(t)^{-1}\hat{X}(t), \hat{N}(t)\right\rangle \hat{N}(t). \tag{12}$$

Now if condition (iii) holds, then (iii)' holds and it implies that

$$\hat{\nabla}_{\dot{\hat{\gamma}}(t)}\hat{X} - U(t)\nabla_{\dot{\gamma}(t)}X = J_{\hat{W}(t)}\hat{X}(t) - \left\langle J_{\hat{W}(t)}\hat{X}(t), \hat{N}(t)\right\rangle \hat{N}(t) = 0,$$

where the last equality follows from the fact that the vector $J_{\hat{W}(t)}\hat{X}(t) = \hat{W}(t)\times\hat{X}(t) \in \mathbb{R}^3$ points in direction parallel to $\hat{N}|_{\hat{\gamma}(t)} \in \mathbb{R}^3$, because $\hat{X}(t), \hat{W}(t) \in T|_{\hat{\gamma}(t)}\hat{M}$. Hence (11) follows.

Conversely, if $\hat{\nabla}_{\dot{\hat{\gamma}}(t)}(UX) - U(t)\nabla_{\dot{\gamma}(t)}X = 0$, for any vector field X along γ and tangent to M we have from (12) that

$$\dot{U}(t)U(t)^{-1}\hat{X}(t) = \left\langle \dot{U}(t)U(t)^{-1}\hat{X}(t), \hat{N}(t)\right\rangle \hat{N}(t)$$

where $\hat{X}(t) := U(t)X(t)$. Fix t and choose $\hat{x}, \hat{y} \in T|_{\hat{x}}\hat{M}$ such that $\hat{x}, \hat{y}, \hat{N}(t)$ is positively oriented orthonormal basis of $\mathbb{R}^3$. Then one may write $V(t) := \dot{U}(t)U(t)^{-1} = \alpha J_{\hat{x}} + \beta J_{\hat{y}} + \gamma J_{\hat{N}(t)}$ and the above formula implies

$$\begin{aligned}\left\langle V(t)\hat{x}, \hat{N}(t)\right\rangle \hat{N}(t) &= V(t)\hat{x} = -\beta\hat{N}(t) + \gamma\hat{y} \\ \left\langle V(t)\hat{y}, \hat{N}(t)\right\rangle \hat{N}(t) &= V(t)\hat{y} = \alpha\hat{N}(t) + \gamma\hat{x},\end{aligned}$$

which both imply that $\gamma = 0$ i.e. $\left\langle V(t), J_{\hat{N}(t)}\right\rangle_{\mathfrak{so}(3)} = 0$. This establishes the condition (iii). □

Hence we have shown that under the correspondence between $Q_{\mathbb{R}^3}$ and Q, the no spinning condition (iii) becomes $\hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(\cdot)X(\cdot)) = A(t)\nabla_{\dot{\gamma}(t)}X$. We have therefore shown the following.

Proposition 4.22 A curve $(\gamma(t), \hat{\gamma}(t), U(t))$ in $Q_{\mathbb{R}^3}$ satisfies the conditions (i)-(iii) if and only if the curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in Q determined by Lemma 4.17 satisfies: For all $t \in [0, 1]$,

(I) No-slipping: $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$.

(II) No-spinning: $A(t)\nabla_{\dot{\gamma}(t)}X(t) = \hat{\nabla}_{\dot{\hat{\gamma}}(t)}(A(t)X(t))$, whenever $X(t)$ is a vector field of M along γ.

The conditions (I) and (II) along with the space $Q(M, \hat{M})$ (defined in the beginning of section 4.2) only depend on the intrinsic Riemannian geometry of $M, \hat{M}$. Hence it is clear that (I) and (II) make sense for any (abstact) oriented Riemannian manifolds (M, g), $(\hat{M}, \hat{g})$ of any dimension, if ∇, $\hat{\nabla}$ are their Levi-Civita connections, respectively. This calls for the following definition.

Definition 4.23 ([2, 4, 7]) Let (M, g), $(\hat{M}, \hat{g})$ be oriented Riemannian manifolds of the same dimension n. We say that M rolls agains $\hat{M}$ along a curve $\gamma : [0, 1] \to M$ *without slipping* and *without spinning* if there exists a curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in Q (which projects to γ on M) such that condition (I) and (II) above are satisfied. In this case we say that $q(t)$ *describes a rolling* of M against $\hat{M}$.

To relate this abstract rolling motion (conditions (I) and (II)) to the distributions $\mathscr{L}_{\overline{\nabla}}$ and $\mathscr{L}_{\mathrm{R}}$ (see Definition 4.4), we make the following obvious observation.

Lemma 4.24 ([4, 7, 13]) For a curve $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ in Q the condition (II) holds if and only if

(II)' $\overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A(\cdot) = 0.$

Here of course $\overline{\nabla}$ is the connection in $T^*M \otimes T\hat{M}$ induced by $\nabla \times \hat{\nabla}$.

Looking at Definition 3.5 and 4.4 we have thus have the following result.

Theorem 4.25 ([2, 4, 7]) Let $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, $t \in [0, 1]$, be a curve in Q. Then

(1) $q(t)$ satisfies the no-spinning condition (II) if and only if

$$\dot{q}(t) \in \mathcal{D}_{\overline{\nabla}}|_{q(t)}, \quad \forall t \in [0, 1].$$

(2) $q(t)$ describes a rolling of M against $\hat{M}$ (i.e. satisfies (I) and (II)) if and only if

$$\dot{q}(t) \in \mathcal{D}_{\mathrm{R}}|_{q(t)}, \quad \forall t \in [0, 1].$$

Proof. (1) If (II) is satisfied by $q(t)$, then it satisfies (II)' and so one has $A(t) = (P^{\overline{\nabla}})_0^t(\gamma, \hat{\gamma})A(0)$. Hence, by Definition 3.5, we have

$$\dot{q}(t) = \dot{A}(t) = \mathscr{L}_{\overline{\nabla}}(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))|_{q(t)} \in \mathcal{D}_{\overline{\nabla}}|_{q(t)}.$$

Conversely, if $\dot{q}(t) \in \mathcal{D}_{\overline{\nabla}}|_{q(t)}$, then Lemma 3.6 (ii) implies that $\dot{A}(t) = \mathscr{L}_{\overline{\nabla}}(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))|_{A(t)}$. On the other hand, by Proposition 3.7

$$\mathscr{L}_{\overline{\nabla}}(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))|_{A(t)} = \dot{A}(t) - \nu(\overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A(\cdot))|_{A(t)}$$

and so $\overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A(\cdot) = 0$ i.e. $q(t)$ satisfies condition (II)', hence (II).

(2) Let $q(t)$ satisfy (I) and (II). Condition (II) implies by the above $\dot{q}(t) = \mathscr{L}_{\overline{\nabla}}(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))|_{q(t)}$ and condition (I) implies that $\dot{\hat{\gamma}}(t) = A(t)\dot{\gamma}(t)$ and so by Definition 4.4

$$\dot{q}(t) = \mathscr{L}_{\overline{\nabla}}(\dot{\gamma}(t), A(t)\dot{\gamma}(t))|_{q(t)} = \mathscr{L}_{\mathrm{R}}(\dot{\gamma}(t))|_{q(t)} \in \mathcal{D}_{\mathrm{R}}|_{q(t)}.$$

Conversely, if $\dot{q}(t) \in \mathcal{D}_{\mathrm{R}}|_{q(t)}$, then since $\mathcal{D}_{\mathrm{R}} \subset \mathcal{D}_{\overline{\nabla}}$, the case (1) shows that $q(t)$ satisfies (II). Moreover, clearly $\dot{q}(t) = \mathscr{L}_{\mathrm{R}}(\dot{\gamma}(t))|_{q(t)} = \mathscr{L}_{\overline{\nabla}}(\dot{\gamma}(t), A(t)\dot{\gamma}(t))$ so by Lemma 3.6 (ii) we obtain

$$(\dot{\gamma}(t), A(t)\dot{\gamma}(t)) = (\pi_{T^*M \otimes T\hat{M}})_* \dot{q}(t) = (\dot{\gamma}(t), \dot{\hat{\gamma}}(t))$$

and so $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$, which is condition (I). $\square$

Remark 4.26 One can easily relax the assumption in Definition 4.23. Indeed, we may assume M and $\hat{M}$ are any manifolds of dimensions n and $\hat{n}$ (possibly $n \neq \hat{n}$) that are equipped with some connections ∇, $\hat{\nabla}$ (for TM and $T\hat{M}$). In this general setting, the definition of the space $Q = Q(M, \hat{M})$ usually does not make sense and it must therefore be replace it with $TM \otimes T\hat{M}$ or some other subbundle of $\pi_{TM \otimes T\hat{M}}$.

However, there is a natural generalization of Q to the case of Riemannian manifolds M, $\hat{M}$ of different dimensions, $n \neq \hat{n}$ (cf. [4]). Indeed, if $n < \hat{n}$, then one defines

$$Q(M, \hat{M}) := \big\{A \in T^*|_x M \otimes T|_{\hat{x}}\hat{M} \mid (x, \hat{x}) \in M \times \hat{M},\ \|AX\|_{\hat{g}} = \|X\|_g\,,\ \forall X \in T|_x M\big\}$$

and if $n > \hat{n}$,

$$Q(M, \hat{M}) := \big\{A \in T^*|_x M \otimes T|_{\hat{x}}\hat{M} \mid (x, \hat{x}) \in M \times \hat{M},\ \|AX\|_{\hat{g}} = \|X\|_g\,,\ \forall X \in (\ker A)^{\perp},\ A \text{ surjective}\big\}.$$

If $n \leq \hat{n}$, then $\dim Q(M, \hat{M}) = n + \hat{n} + n\hat{n} - \frac{n(n-1)}{2}$ and if $n \geq \hat{n}$, then $\dim Q(M, \hat{M}) = n + \hat{n} + n\hat{n} - \frac{\hat{n}(\hat{n}-1)}{2}$. It is an elementary fact that the map

$$Q(M, \hat{M}) \to Q(\hat{M}, M); \quad (x, \hat{x}; A) \mapsto (\hat{x}, x; A^T).$$

is diffeomorphism, so only one of the above definitions is actually needed. Here A^T is the transpose of A with respect to the metrics $g, \hat{g}$. The distributions $\mathcal{D}_{\overline{\nabla}}$ and $\mathcal{D}_{\mathrm{R}}$ restrict to distributions on $Q(M, \hat{M})$ of ranks $2n$ and n, respectively. This follows immediately from the proof of Lemma 4.9.

Also, both above definitions make sense if $n = \hat{n}$ and give two-sheeted covering spaces (having 2 connected components) of the state space Q as defined in Section 4.2, when M, $\hat{M}$ are oriented. The reason is that in that section, we also imposed the condition that A be orientation preserving.

For given Riemannian manifolds (M, g), $(\hat{M}, \hat{g})$ of the same dimension n, the problem of complete controllability of the *rolling problem* consists in determining when for all $q_0, q_1 \in Q$ one can find a curve $q(t)$, $t \in [0, 1]$, in Q that describes rolling of M agains $\hat{M}$, in the sense of Definition 4.23, and satisfies $q(0) = q_0$, $q(1) = q_1$. By Theorem 4.25 this problem is equivalent to the study of the controllability of the distribution $\mathcal{D}_{\mathrm{R}}$ i.e. when is it true that $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = Q$ for some (and hence every) $q_0 \in Q$.

The distribution $\mathcal{D}_{\mathrm{R}}$ being a subdistribution of $\mathcal{D}_{\overline{\nabla}}$, one has $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) \subset \mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0)$ for all $q_0 \in Q$. Hence, if $\mathcal{D}_{\overline{\nabla}}$ is not completely controllable ($\mathcal{O}_{\mathcal{D}_{\overline{\nabla}}}(q_0) \neq Q$), then $\mathcal{D}_{\mathrm{R}}$ is not completely controllable. Therefore the study of $\mathcal{D}_{\overline{\nabla}}$ and its orbits gives necessary conditions for the controllability of $\mathcal{D}_{\mathrm{R}}$. In particular, Proposition 4.14 shows that if (M, g) and $(\hat{M}, \hat{g})$ are both Riemannian products, the rolling problem is not completely controllable.

Besides these easy remarks, the formulas for Lie brackets of vector fields in Q with respect to the splitting $TQ = \mathcal{D}_{\overline{\nabla}} \oplus V(\pi_Q)$ as given in sections 3.4 and 4.1, turn out to be very useful when computing the (generally infinite dimensional) Lie algebraic structure of $\mathcal{D}_{\mathrm{R}}$ and determining whether $\mathcal{D}_{\mathrm{R}}$ is controllable or not.

Acknowledgements

The work of the author is part of his PhD thesis and is supported by Finnish Academy of Sciences and Letters, KAUTE foundation and Institut français de Finlande. The author also wants to thank his advisor Y. Chitour for having suggested the problem.

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